GENERATING RELATIONS AND TRANSFORMATIONS OF MULTIPLE GAUSSIAN HYPERGEOMETRIC SERIES

ABSTRACT

THESIS

SUBMITTED FOR THE AWARD OF THE DEGREE OF

Doctor of Philosophy

IN

MATHEMATICS

BY

MAGED JUMAN AWADH BIN-SAAD

Under the Supervision of

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DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
ALIGARH MUSLIM UNIVERSITY
ALIGARH (INDIA)

2000

PREFACE

The importance of working with multiple hypergeometric series of any kind stems from the fact that the majority of special functions are simply special cases of them, and thus each generating relation or transformation formula developed for the multiple hypergeometric series becomes a master formula from which a large number of relations for other special functions can be deduced. Also, the use of multiple hypergeometric series often facilitates the analysis by permitting complex expressins to be represented more simply in terms of some multivariable function.

Hypergeometric series in one or more variables occur frequently in a wide variety of problems in physics, mathematics, statistics, engineering sciences and operations research. Motivated by a vast field of applications of these multiple hypergeometric series, many research workers obtained generating relations, expansions, transformations and reduction formulae for multivariable functions.

In view of the growing importance of generating funtions, this thesis contains certain double generating functions which are bilinear, bilateral and partly bilateral and partly unilateral for a fairly wide variety of special functions and polynomials in one, two and more variables. Such generating relations are obtained by using series rearrangement and integral transforms techniques. Some transformation and reduction formulae for double and triple hypergeometric series are also presented and various special cases are deduced.

A number of known results follow as special cases of our findings and may more new results can be obtained by appropriately specializing the coefficient.

This thesis comprises of seven chapters. A brief summary of the problem is presented at the begining of each chapter and then chapter is divided into a number of sections. Definitions and equations have been numbered chapterwise. The equations are numbered in such a way that, when read as decimals they stand in their proper order. For example, the bracket (a. b. c) specified the result, in which last decimal place represents the equation number, the middle one the section and the first indicates the chapter to which it belongs. The thesis has been made self contained by the inclusion, in chapter I, of the brief treatment of the theory, definitions and notations of special functions with their convergence conditions.

Chapter II, begins by introducing a new class of interesting relation involving Shively's generalized hypergeometric function $S_n^{\alpha}(x)$ [78]. Motivated by results of a number of workers including (for example) Pathan and Yasmeen ([68] and [69]), Kamarujjama et al. [46] and Srivastava et al. [96], we have obtained a general theorem on partly bilateral and partly unilateral generating functions involving multiple series with essentially arbitrary coefficients. A number of known results of Pathan and Yasmeen [69], Exton ([26] and [31]), Kamarujjama et al. [46] and Goyal and Laddha [35] are special cases of our theorem. Besides treating these results, certain correct forms of results of Kamarrujjama et al. [46] also follow as consequences of our theorem.

In chapter III, we obtained three general theorems on bilinear and bilateral generating functions involving multiple series with essentially arbitrary coefficients. By appropriately specializing these coefficients, a number of known results of Mathur [61], Chaudhary [16], Saran [76], Srivastava [87], Manocha [58], Manocha and Sharma [59], Sharma and Mittal [79], Chaundy [17], Gupta [38], Brafman [8] and Saxena [77] are shown to follow as applications of these theorems. More interestingly, the main generating relation proven recently by Chaudhary [16] is also corrected here.

In chapter IV, we have obtained generating functions for Gauss hypergeometric function ${}_2F_1$ and generalized hypergeometric function ${}_AF_B$ and further generalized it to n-variables $F^{A:B';\ldots\ldots;B^{(n)}}_{C:D';\ldots\ldots;D^{(n)}}[x_1,\ldots\ldots,x_n]$, which are double generating functions of single hypergeometric function. On specializing our main results, a number of double generating functions for polynomials of Legendre and Jacobi and functions of several variables of Kampe' de Fe'riet, Lauricella and Appell are obtained. These generating functions include some of the results given earlier by Exton [32], Srivastava ([85] and [87]), Srivastava and Manocha [94] and Chaundy [94]. Moreover, certain correct form of known result of Exton [32] also follow as special case of our main generating relation.

In chapter V , an attempt has been made to establish Bessel polynomials in several variables $y^{(\alpha_1,\ldots\ldots,\alpha_n;\beta)}_{m_1,\ldots\ldots,m_n}(x_1,\ldots\ldots,x_n)$, which provide multivariable generalization of known Bessel polynomials of Al-Salam [3], Chatterjea [15], Krall and Frink [53] and Mumtaz and Khursheed [63]. A number of special cases are discussed and also we have obtained a partly bilateral and partly unilateral generating functions for the Bessel polynomials $y_{m_1,\ldots,m_n}^{(\alpha_1,\ldots,\alpha_n;\beta)}(x_1,\ldots,x_n)$, Lauricella $F_A^{(n)}$ and generalized Horn's function ${}^{(k)}H_4^{(n)}$ of n-variables.

Chapter VI provides us some unification and representation of Voigt functions K(x,y) and L(x,y) which play an important role in serveral diverse fields of physics such as astrophysical spectroscopy, emission, absorption and transfer of radiation in heated atmosphere and the theory of neutron reactions. We have derived several representations of these functions in terms of series and integrals which are specially useful in situations when the parameters and variables take on particular values. The generating functions (partly bilateral and partly unilateral) will follow as natural consequences of integral representations for analytic evaluations and uses. Many interesting special cases of our generating relations are also discussed.

Finally, in Chapter VII, an integral formula involving the product of three Bessel functions of different order and arguments plays a key role in obtaining transformation and reduction formulae of Srivastava's triple hypergeometric series $F^{(3)}$, Lauricella's series $F_C^{(3)}$ and Kampe' de Fe'riet function $F_{C:D;D'}^{A:B';B}$. Also we obtained some special cases which reduce $F^{(3)}$ to $F_A^{(3)}$, $F_C^{(3)}$, H_4 and $H_4^{(p)}$, $F_{0:3;1}^{2:2;0}$ to ψ_1 and ${}_0F_1$, $F_{0:3;1}^{2:1;0}$ to $F_A^{(3)}$ and $F_C^{(3)}$ and $F_C^{(3)}$ to F_4 and $({}_2F_1+{}_2F_1)$. Some known results of Pathan and Khan [66], Erde'lyi [22] and Waston [98] are deduced as special cases.

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Certificate

This is to certify that the contents of this thesis entitled "Generating Relations and Transformations of Multiple Gaussian Hypergeometric Series" is the original research work of Mr. Maged Juman Bin-Saad carried out under my supervision. He has fulfilled the prescribed conditions given in the ordinances and regulations of Aligarh Muslim University, Aligarh.

I further certify that the work has not been submitted either partly or fully to any other University or Institution for the award of any degree.

Countersigned

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CHAPTER-I

INTRODUCTION, DEFINITIONS AND NOTATIONS

1.1 INTRODUCTION

A wide range of problems exists in classical and quantum physics, engineering and applied mathematics in which special functions arise.

The procedure followed in most investigations on these topics (e.g., quantum mechanics, electrodynamics, modern physics, classical mechanics, etc.) is to formulate the problem as a differential equation that is related to one of several special differential equation (Hermite's, Bessel's, Laguerre's, Legendre's, etc.). The corresponding special functions are then introduced as solution with some discussion of recursion formulae, orthogonality relations, and other properties as appropriate. Each special function can be defined in variety of ways and different researchers may choose different definitions (Rodrigues formulae, generating functions, contour integrals, etc.). Most of the special functions have a common root in their relation to the hypergeometric function.

The aim of the present chapter is to introduce the several classes of special functions which occur rather more frequently in the study of generating functions and transformations and needed for presentation of the subsequent chapters. First, we recall some definitions and identities involving Pochhammer symbol $(\lambda)_n$, Gamma function Γz and related functions.

The Gamma Function

The Gamma function crops up repeatedly in applied mathematical analysis. It has several equivalent definitions, most of which are due to Euler.

We follow Euler in defining the function z by

$$\Gamma z = \int_0^\infty t^{z-1} e^{-t}\, dt, \quad Re(z)>0. \qquad (1.1.1)$$

Upon integration by parts, definition (1.1.1) yields the recurrence relation

$$\Gamma(z+1) = z\,\Gamma z \qquad (1.1.2)$$

From the relations (1.1.1) and (1.1.2) it follows that :

$$\Gamma z = \begin{cases} \int_0^\infty t^{z-1} e^{-t}\, dt, & Re(z)>0. \\ \dfrac{\Gamma(z+1)}{\Gamma z}, & Re\,(z)<0,\ z\neq 0,\ -1,\ -2, \ldots\ldots\ldots. \end{cases} \qquad (1.1.3)$$

The recurrence relation (1.1.2) yields the useful result

$$\Gamma(n+1) = n! \quad , \quad n=0,1,2, \ldots\ldots\ldots\ldots.$$

which shows that the Gamma function Γz is a generalization of the factorial function

$$n! = \int_0^\infty t^n\, e^{-t}\, dt, \quad n = 0,\ 1,\ 2 \ldots\ldots\ldots \qquad (1.1.4)$$

The Beta function

We define the Beta function by

$$B(x,y) = \int_0^1 t^{x-1}\, (1-t)^{y-1}\, dt, \qquad (1.1.5)$$

$$(\, Re\,(y)>0,\ Re\,(x)>0\,).$$

The Beta function is closely related to the Gamma function, in fact, we have

$$B(x,y) = \frac{\Gamma x \; \Gamma y}{\Gamma x+y} \quad , \; x\,,\, y \neq 0, -1, -2...... \tag{1.1.6}$$

Thus we may write by analogy with (1.1.3), that

$$B\,(x,\,y) = \begin{cases} \int_0^1 t^{x-1} \;(1-t)^{y-1}\, dt\,,\; Re(x)>0,\; Re\,(y)>0, \\ \\ \dfrac{\Gamma x \; \Gamma y}{\Gamma x+y} \qquad\qquad , \; x,\, y \neq 0, -1, -2...... \end{cases} \tag{1.1.7}$$

The Pochhammer Symbol

The Pochhammer symbol $(\lambda)_n$ is defined by

$$(\lambda)_n = \begin{cases} 1 & ,\text{ if } n=0 \\ \\ \lambda\,(\lambda+1)..........(\lambda+n-1), & \text{if } n=1,\,2,\,3,\,......... \end{cases} \tag{1.1.8}$$

In terms of Gamma function, we have

$$(\lambda)_n = \frac{\Gamma \lambda+n}{\Gamma \lambda} \quad , \; \lambda \neq 0\;,\; -1,\, -2,......... \tag{1.1.9}$$

Furthermore, the binomial coefficient may be expressed as

$$\binom{\lambda}{n} = \frac{\Gamma \lambda+1}{n!\,\Gamma \lambda-n+1} \; . \tag{1.1.10}$$

Equation (1.1.9) yields

$$(\lambda)_{m+n} = (\lambda)_m \;(\lambda+m)_n \tag{1.1.11}$$

which in conjuction with the relation

$$(\lambda)_{-n} = \frac{(-1)^n}{(1-\lambda)_n}\ , \tag{1.1.12}$$

gives

$$(\lambda)_{n-m} = \frac{(-1)^m\ (\lambda)_n}{(1-\lambda-n)_m}\ ,\ 0\leq m \leq n\ . \tag{1.1.13}$$

For $\lambda=1$, relation (1.1.13) reduces to

$$(n-m)! = \frac{(-1)^m\ n!}{(-n)_m}\ ,\ 0 \leq m \leq n \tag{1.1.14}$$

which may alternatively be written in the form :

$$(-n)_m \quad \begin{cases} \dfrac{(-1)^m\, n!}{(n-m)!}\ , & 0\leq m \leq n \\ 0\ , & m > n. \end{cases} \tag{1.1.15}$$

Another useful relation of the Pochhammer symbol $(\lambda)_n$ is included in Gauss's multiplication theorem :

$$(\lambda)_{mn} = m^{mn} \prod_{j=1}^{m} \left(\frac{\lambda+j-1}{m}\right)_n\ ,\ n=0,\ 1,\ 2,\ \ldots\ldots, \tag{1.1.16}$$

where m is positive integer. For m=2, equation (1.1.16) reduces to Legendre duplication formula:

$$(\lambda)_{2n} = 2^{2n}\ (\tfrac{1}{2}\lambda)_n\ \ (\tfrac{1}{2}\lambda+\tfrac{1}{2})_n\ ,\quad n=0,\ 1,\ 2,\ldots. \tag{1.1.17}$$

In particular we have

$$(2n)! = 2^{2n}\ (\tfrac{1}{2})_n\ \ n! \quad \text{and} \quad (2n+1)! = 2^{2n}\ (3/2)_n\ \ n! \tag{1.1.18}$$

1.2. GAUSS HYPERGEOMETRIC FUNCTION

A function given by

$$ {}_2F_1\,(a,\ b;\ c;\ z) = 1 + \frac{a.b}{1.c}\,z + \frac{a(a+1)\ b(b+1)}{1.2.c\ (c+1)}\,z^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots $$

$$ = \sum_{n=0}^{\infty} \frac{(a)_n\ (b)_n}{(c)_n}\ \frac{z^n}{n!},\ c \neq 0,\ -1,\ -2, \ldots. \qquad (1.2.1) $$

is known as Gauss's hypergeometric function.

The special case $a = c$ and $b=1$ or $a=1$ and $b=c$ yields the elementary geometric series

$$ \sum_{n=0}^{\infty} z^n = 1+z+z^2+\ldots\ldots\ldots+z^n+\ldots\ldots., \qquad (1.2.2) $$

hence the term hypergeometric. Hypergeometric series or more precisely Gauss series is due to Carl Friedrich Gauss (1777-1855) who in year 1812 introduced this series into analysis and gave the F-notation for it.

If either of the parameters a or b is negative integer m, then in this case (1.2.1) reduces to hypergeometric polynomial defined by

$$ F(-m,\ b;\ c;\ z) = \sum_{n=0}^{m} \frac{(-m)_n\ (b)_n}{(c)_n}\ \frac{z^n}{n!}, \qquad (1.2.3) $$

$$ (-\infty < z < \infty). $$

This series given by (1.2.1) converges absolutely within the limit circle $|z|<1$ and when $|z| =1$, provided that $Re(c-a-b)>0$ for $z = 1$ and $Re(c-a-b)> -1$ for $z = -1$.

1.3 GENERALIZED HYPERGEOMETRIC FUNCTION

The hypergeometric function defined in equation (1.2.1) can be generalized in an obvious way

$$
{}_pF_q\left[\begin{matrix} a_1, a_2, \ldots\ldots, a_p; \\ b_1, b_2, \ldots\ldots, b_q; \end{matrix}\; z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n\,(a_2)_n \ldots\ldots (a_p)_n}{(b_1)_n\,(b_2)_n \ldots\ldots (b_q)_m}\,\frac{z^n}{n!}
$$

$$
= \sum_{n=0}^{\infty} \frac{((a_p))_n}{((b_q))_n}\,\frac{z^n}{n!} \tag{1.3.1}
$$

where, as usual

$$
((a_p))_n = (a_1)_n \ldots\ldots (a_p)_n = \prod_{j=1}^{p} (a_j)_n \tag{1.3.2}
$$

with similar interpretation for $((b_q))_n$.

Here $(\lambda)_n$ is Pochhammer symbol (1.1.8) , p and q are positive integers or zero and we assume that the variable z , the numerator parameters $a_1, \ldots, a_p$, and the denominator parameters $b_1, \ldots, b_q$ take on complex values provided that

$$
b_j \neq 0, -1, -2, \ldots\ldots, \quad j=1,2,\ldots\ldots,q\,. \tag{1.3.3}
$$

On other hand, assuming that none of the numerator parameters is zero or a negative integer (otherwise series will not converge), we have the following cases for ${}_pF_q$ series in (1.3.1)

(i) converges for $|z| < \infty$ if $p \leq q$,

(ii) converges for $|z| < 1$ if $p=q+1$, and

(iii) diverges for all z , $z \neq 0$, if $p > q+1$ and converges only when $z=0$.

Furthermore, if we set

$$w = \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i \quad ,$$

the ${}_pF_q$ series, with p=q+1, is

(i) absolutely convergent for $|z|=1$, if (w)>0,

(ii) conditionally convergent for $|z|=1$, $z\neq 1$, if $-1<\mathrm{Re}(w)\leq 0$,

and

(iii) divergent for $|z|=1$ if $\mathrm{Re}\,(w)\leq -1$.

When p=q=1, (1.3.1) reduces to the confluent hypergeometric series ${}_1F_1$ named as Kummer's series [54]. When p=2 and q=1, (1.3.1) reduces to ordinary hypergeometric function ${}_2F_1$ of second order given by equation (1.2.1).

1.4 APPELL'S FUNCTIONS

In addition to increasing the number of parameters, hypergeometric functions may be generalized along the lines of increasing the number of variabels. Appell [4,p.296(1)] defined the four hypergeometric functions of two varaibles which follow :

$F_1\,[a, b, b'; c; x, y]$

$$= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}\,(b)_m\,(b')_n}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{1.4.1}$$

$$\max\{\,|x|, |y|\,\} < 1;$$

F_2 [a, b, b'; c, c'; x, y]

$$= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b)_m (b')_n}{(c)_m (c')_n} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{1.4.2}$$

$$|x| + |y| < 1;$$

F_3 [a, a', b, b'; c; x, y]

$$= \sum_{m,n=0}^{\infty} \frac{(a)_m (a')_n (b)_m (b')_n}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{1.4.3}$$

$$\max\{|x|, |y|\} < 1;$$

F_4 [a, b; c, c'; x, y]

$$= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b)_{m+n}}{(c)_m (c')_n} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{1.4.4}$$

$$\sqrt{|x|} + \sqrt{|y|} < 1.$$

The functions F_1, F_2, F_3 and F_4 given above are all generalization of the Gauss hypergeometric functions ${}_2F_1$. Here as usual, the denominator parameters c and c' are neither zero nor a negative integers . The standard work on the theory of Appell series is the monograph by Appell and Kampe' de Fe'riet [5]. See Erde'lyi et al [22, p.222-245] for a review of the subsequent work on the subject; see also Slater [81,Chapter 8] and Exton [27, p.23-28].

1.5 HUMBERT'S FUNCTIONS

In 1920, Humbert [44] has studied seven confluent forms of the four Appell functions and denoted them by

$$\Phi_1, \quad \Phi_2, \quad \Phi_3, \quad \psi_1, \quad \psi_2, \quad \Xi_1, \Xi_2,$$

four of them are given below [22] :

$$\Phi_1[\alpha, \beta; \gamma; x, y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta)_m}{(\gamma)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{1.5.1}$$

$$|x| < 1, \quad |y| < \infty;$$

$$\Phi_2[\beta;\beta'; \gamma; x, y] = \sum_{m,n=0}^{\infty} \frac{(\beta)_m (\beta')_n}{(\gamma)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{1.5.2}$$

$$|x| < \infty, |y| < \infty;$$

$$\psi_1[\alpha, \beta; \gamma; \gamma'; x, y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta)_m}{(\gamma)_m (\gamma')_n} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{1.5.3}$$

$$|x| < 1, |y| < \infty;$$

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$$|x| < \infty, |y| < \infty.$$

1.6 HORN'S FUNCTIONS

Other hypergeometric functions of two variables has been defined by Horn. Two of them are given below [94, p.56-57].

$$H_1[a, b, c; d; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m-n} (b)_{m+n} (c)_n}{(d)_m} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{1.6.1}$$

$$|x| < r, |y| < s, 4rs = (s-1)^2 ;$$

$$H_4[a, b; c, d; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n} (b)_n}{(c)_m (d)_n} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{1.6.2}$$

$$|x| < r, |y| < s, 4r = (s-1)^2 .$$

1.7 KAMPE' DE FE'RIET'S FUNCTION

In an attempt to generalize the four Appell function F_1 to F_4, Kampe' de Fe'riet [45] defined a general hypergeometric series in two variables (see [5,p.150(29)]). Kampe' de Fe'riet's function is denoted by $F^{A:B;D}_{E:G;H}$ and is defined as follows [28,p.24]

$$F^{A:B;D}_{E:G;H}\left[\begin{matrix}(a) : (b); (d); \\ & x, y \\ (e) : (g); (h);\end{matrix}\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{((a))_{m+n} ((b))_m ((d))_n}{((e))_{m+n} ((g))_m ((h))_n} \frac{x^m}{m!} \frac{y^n}{n!} \tag{1.7.1}$$

where for convergence,

(i) $A+B \leq E+G$, $A+D \leq E+H$ for $\max\{ |x| , |y| < \infty\}$,

(ii) $A+B = E+G+1$, $A+D=E+H+1$, and

$$\begin{cases} |x|^{1/(A-E)} + |y|^{1/(A-E)} < 1, & \text{if } A > E, \\ \max\{|x|, |y|\} < 1, & \text{if } A \leq E. \end{cases}$$

1.8 SRIVASTAVA'S TRIPLE HYPERGEOMETRIC FUNCTION

In 1967, a unification of Lauricella fourteen hypergeometric functions $F_1, \ldots, F_{14}$ of three variables [56], and Srivastava's three additional functions H_A, H_B, H_C, [82], was introduced by Srivastava (see e.g. [83,p.428] and [94,p.69]) in the form of a general triple hypergeometric series $F^{(3)}[x,y,z]$ defined as

$$F^{(3)}[x,y,z] = F^{(3)}\left[\begin{matrix} (a)::(b);\ (b');\ (b'') : (c);\ (c');\ (c''); \\ \\ (e)::(g);\ (g');\ (g'') : (h);\ (h');\ (h''); \end{matrix} \quad x,y,z\right]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{((a))_{m+n+p}((b))_{m+n}((b'))_{n+p}((b''))_{p+m}((c))_m((c'))_n((c''))_p\, x^m\, y^n\, z^p}{((e))_{m+n+p}((g))_{m+n}((g'))_{n+p}((g''))_{p+m}((h))_m((h'))_n((h''))_p\, m!\, n!\, p!} \qquad (1.8.1)$$

For convergence of the series (1.8.1), see [94, p.70 (41)] .

1.9 PATHAN'S FUNCTION

In 1979, a general quadruple hypergeometric series $F_p^{(4)}$ was considered by Pathan [64,p.172(1.2)], (see also [72]) in the form

$$F_P^{(4)}\left[\begin{matrix} (a_A)::(b_B);(d_D);(e_E);(f_F):(g_G);(k_K);(\ell_L);(m_M):(h_H);(u_U);(v_V);(n_N); \\ \\ (a'_{A'})::(b'_{B'});(d'_{D'});(e'_{E'});(f'_{F'}):(g'_{G'});(k'_{K'});(\ell'_{L'});(m'_{M'}):(h'_{H'});(u'_{U'});(v'_{V'});(n'_{N'}); \end{matrix} \quad x,y,z,w\right]$$

$$= \sum_{p,q,s,j=0}^{\infty} \frac{((a_A))_{p+q+s+j}((b_B))_{p+q+s}((d_D))_{q+s+j}((e_E))_{s+j+p}(f_F))_{j+p+q}((g_G))_{p+q}((k_K))_{q+s}((\ell_L))_{s+j}}{((a'_{A'}))_{p+q+s+j}((b'_{B'}))_{p+q+s}((d'_{D'}))_{q+s+j}((e'_{E'}))_{s+j+p}(f'_{F'}))_{j+p+q}((g'_{G'}))_{p+q}((k'_{K'}))_{q+s}((\ell'_{L'}))_{s+j}}$$

$$\frac{((m_M))_{j+p}((h_H))_p((u_U))_q((v_V))_s((n_N))_j \; x^p \; y^q \; z^s \; w^j}{((m'_{M'}))_{j+p}((h'_{H'}))_p((u'_{U'}))_q((v'_{V'}))_s((n'_{N'}))_j \; p! \; q! \; s! \; j!} \qquad (1.9.1)$$

It being understood that $|x|$, $|y|$, $|z|$ and $|w|$ are sufficiently small to ensure the convergence of the concerned quadruple series.

By suitable adjustment of parameters and variables in $F_P^{(4)}$, we can easily find that $F_P^{(4)}$ is unification of triple hypergeometric series $F^{(3)}$ of Srivastava, Exton's ${}^{(0)}_{(1)}E_D^{(4)}$, ${}^{(1)}_{(1)}E_D^{(4)}$, ${}^{(3)}_{(1)}E_D^{(4)}$, ${}^{(4)}_{(1)}E_D^{(4)}$, ${}^{(0)}_{(2)}E_D^{(4)}$, ${}^{(1)}_{(2)}E_D^{(4)}$, ${}^{(3)}_{(2)}E_D^{(4)}$, ${}^{(4)}_{(2)}E_D^{(4)}$, K_{11}, K_{15}, Lauricell's $F_A^{(4)}$, $F_B^{(4)}$, $F_C^{(4)}$, $F_D^{(4)}$, Erde'lyi's $\Phi_2^{(4)}$, Chandel's ${}^{(0)}_{(1)}E_C^{(4)}$, ${}^{(1)}_{(1)}E_C^{(4)}$, ${}^{(4)}_{(1)}E_C^{(4)}$, ${}^{(3)}_{(1)}E_C^{(4)}$, and Humbert's $\psi_2^{(4)}$.

For the definitions of the above functions we refer the books of Exton [27] and Srivastava and Manocha [94].

1.10 LAURICELLA'S FUNCTIONS OF n-VARIABLES

In 1893, Lauricella [56] further generalized the four Appell functions $F_1,\ldots,F_4$ to functions of n-variables and defined his functions as follows [94,p.60]

$$F_A^{(n)}[a, b_1,\ldots,b_n; c_1,\ldots,c_n; x_1,\ldots,x_n]$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}(b_1)_{m_1}\ldots\ldots(b_n)_{m_n}}{(c_1)_{m_1}\ldots\ldots(c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}\ldots\ldots\frac{x_n^{m_n}}{m_n!}, \qquad (1.10.1)$$

$$|x_1|+\ldots+|x_n| < 1;$$

$$F_B^{(n)}[a_1,\ldots,a_n, b_1,\ldots b_n; c; x_1,\ldots,x_n]$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a_1)_{m_1}\ldots(a_n)_{m_n}(b_1)_{m_1}\ldots\ldots(b_n)_{m_n}}{(c)_{m_1+\ldots\ldots+m_n}} \frac{x_1^{m_1}}{m_1!}\ldots\ldots\frac{x_n^{m_n}}{m_n!}, \tag{1.10.2}$$

$$\max\{|x_1|,\ldots,|x_n|\} < 1;$$

$$F_C^{(n)}[a, b; c_1,\ldots,c_n; x_1,\ldots,x_n]$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}(b)_{m_1+\ldots m_n}}{(c_1)_{m_1}\ldots\ldots(c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}\ldots\ldots\frac{x_n^{m_n}}{m_n!}, \tag{1.10.3}$$

$$\sqrt{|x_1|}+\ldots+\sqrt{|x_n|} < 1;$$

$$F_D^{(n)}[a, b_1,\ldots, b_n; c; x_1,\ldots,x_n]$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}(b_1)_{m_1}\ldots(b_n)_{m_n}}{(c)_{m_1+\ldots\ldots+m_n}} \frac{x_1^{m_1}}{m_1!}\ldots\ldots\frac{x_n^{m_n}}{m_n!}, \tag{1.10.4}$$

$$\max\{|x_1|,\ldots\ldots,|x_4|\} < 1.$$

In particular, we have,

$$F_A^{(2)} = F_2,\ F_B^{(2)} = F_3,\ F_C^{(2)} = F_4 \text{ and } F_D^{(2)} = F_1 \tag{1.10.5}$$

and

$$F_A^{(1)} = F_B^{(1)} = F_C^{(1)} = F_D^{(1)} = {}_2F_1.$$

Two important confluent hypergeometric functions of n-variables are the functions $\Phi_2^{(n)}$ and $\psi_2^{(n)}$ defined by [94,p.62(10) and (11)].

$\Phi_2^{(n)}[b_1,\ldots\ldots,b_n; c; x_1,\ldots\ldots,x_n]$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(b_1)_{m_1}\ldots\ldots(b_n)_{m_n}}{(c_1)_{m_1+\ldots\ldots m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\cdots\frac{x_n^{m_n}}{m_n!} \; ; \tag{1.10.6}$$

and

$\psi_2^{(n)}[a; c_1,\ldots,c_n; x_1,\ldots,x_n]$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}}{(c_1)_{m_1}\cdots\cdots(c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\cdots\frac{x_n^{m_n}}{m_n!} \tag{1.10.7}$$

Clearly, we have

$$\Phi_2^{(2)} = \Phi_2 \text{ and } \psi_2^{(2)} = \psi_2,$$

where ϕ_2 and ψ_2 are Humbert's confluent forms of two variables.

1.11 GENERALIZED LAURICELLA FUNCTION

A further generalization of Kampe' de Fe'riet function (1.7.1) *and* Lauricella functions $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$ is due to Srivastava and Daost (cf. [90] and [92]) who indeed defined an extension of Wright's ${}_p\psi_q$ function [94,p.50(21)] in two variables.

The generalized Lauricella function is defined as follows :

$$F^{A:B';\ldots;B^{(n)}}_{C:D';\ldots;D^{(n)}}\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}]; \\ [(c):\psi',\ldots,\psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\; x_1,\ldots,x_n\right]$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{((a))_{m_1\theta'_j+\ldots+m_n\theta_j^{(n)}}((b'))_{m_1\phi'_j}\cdots((b^{(n)}))_{m_n\phi_j^{(n)}}\, x_1^{m_1}\ldots x_n^{m_n}}{((c))_{m_1\psi'_j+\ldots+m_n\psi_j^{(n)}}((d'))_{m_1\delta'_j}\cdots((d^{(n)}))_{m_n\delta_j^{(n)}}\; m_1!\ldots m_n!} \tag{1.11.1}$$

The coefficients

$$\theta_j^{(k)},\ j=1,\ldots,A;\ \phi_j^{(k)},\ j=1,\ldots,B^{(k)};\ \psi_j^{(k)},\ j=1,\ldots,C;$$

$$\delta_j^{(k)},\ j=1,\ldots,\ D^{(k)};\ \phi_j^{(k)},\quad \forall\in\{1,\ldots,n\}$$

are real and positive, and $(b^{(k)})$ abbreviates the array of $B^{(k)}$ parameters

$$b_j^{(k)},\ j=1,\ldots.,B^{(k)};\ \forall\ k\in\{1,\ldots,\ n\},$$

with similar interpretations for $(d^{(k)})$, k=1, ...,n; etc. A detailed discussion of the conditions of convergence of the multiple series occuring in (1.11.1), is given in paper of Srivastava and Daoust [92].

If the positive constants θ's, ψ's, ϕ's and δ's are all chosen as unity, then (1.11.1) reduces to the generalized Kampe' de Fe'riet function given by Karlsson [47] in its more general form

$$F_{C:D';\ldots;D^{(n)}}^{A:B';\ldots;B^{(n)}}\left[\begin{matrix}(a):(b');\ldots\ldots\ldots;(b^{(n)});\\(c):(d');\ldots\ldots\ldots;(d^{(n)});\end{matrix}\ z_1,\ldots,z_n\right]$$

$$=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{((a))_{m_1+\ldots m_n}((b'))_{m_1}\ldots((b^{(n)}))_{m_n}}{((c))_{m_1+\ldots+m_n}((d'))_{m_1}\ldots((d^{(n)}))_{m_n}}\frac{x_1^{m_1}}{m_1!}\ldots\ldots\frac{x_n^{m_n}}{m_n!}\qquad(1.11.2)$$

Clearly, we have

$$F_{0:1;\ldots;1}^{1:1;\ldots;1}=F_A^{(n)}\qquad F_{1:0;\ldots;0}^{0:2;\ldots;2}=F_B^{(n)}$$

$$(1.11.3)$$

$$F_{0:1;\ldots;1}^{2:0;\ldots;0}=F_C^{(n)}\qquad F_{1:0;\ldots;0}^{1:1;\ldots;1}=F_D^{(n)}.$$

1.12 GENERALIZED HORN FUNCTION ${}^{(k)}H_4^{(n)}$

A generalized Horn's function ${}^{(k)}H_4^{(n)}$ was introduced by Exton [27,p.97] in the form of multiple series :

$${}^{(k)}H_4^{(n)}\ [a, b_{k+1}, \ldots\ldots, b_n; c_1, \ldots\ldots, c_n;\ x_1, \ldots, x_n]$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{2m_1+\ldots+2m_k+m_{k+1}+\ldots+m_n} (b_{k+1})_{m_{k+1}} \ldots (b_n)_{m_n}}{(c_1)_{m_1} \ldots\ldots\ldots (c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!} \ldots\ldots \frac{x_n^{m_n}}{m_n!} \quad (1.12.1)$$

and the region of convergence is given by [93,p.267(8)]

$$\{2(\sqrt{r_1} + \ldots + \sqrt{r_k}) + r_{k+1} + \ldots\ldots + r_n < 1\}, \qquad (1.12.2)$$

where $r_1, \ldots, r_n$ denotes the absolute values of the variables.

We note the following reducible cases of (1.12.1):

(i) $${}^{(1)}H_4^{(1)}\ [a; c; x] = {}_2F_1\ [\tfrac{1}{2}a, \tfrac{1}{2}a+\tfrac{1}{2}; c; 4x] \qquad (1.12.3)$$

where ${}_2F_1$ is Gauss function defined by (1.2.1).

(ii) $${}^{(1)}H_4^{(2)}\ [a, b; c_1, c_2; x, y] = H_4\ [a, b; c_1, c_2, x, y]\ . \qquad (1.12.4)$$

(iii) $${}^{(1)}H_4^{(3)}\ [a, b, d; c_1, c_2, c_3; x, y, z]$$

$$= H_4^{(p)}\ [a, b, d; c_1, c_2, c_3; x, y, z]\ , \qquad (1.12.5)$$

where H_4 is Horn function defined by (1.6.2) and $H_4^{(p)}$ is Horn function of three variables defined as [48, p.85(1.1)] :

$$H_4^{(p)}[a, b, d; c_1, c_2, c_3; x, y, z]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n+p}(b)_n (d)_p}{(c_1)_m (c_2)_n (c_3)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} . \tag{1.12.6}$$

(iv) $${}^{(2)}H_4^{(2)}[a; c_1, c_2; x, y] = F_4[\tfrac{1}{2}a, \tfrac{1}{2}a+\tfrac{1}{2}; c_1, c_2; 4x, 4y] \tag{1.12.7}$$

where F_4 is Appell's function (cf. (1.4.3)).

(v) $${}^{(0)}H_4^{(n)}[a, b_1,\ldots,b_n; c_1,\ldots,c_n; x_1,\ldots,x_n]$$

$$= F_A^{(n)}[a, b_1,\ldots,b_n; c_1,\ldots,c_n; x_1,\ldots,x_n]. \tag{1.12.8}$$

(vi) $${}^{(k)}H_4^{(k)}[a; c_1,\ldots,c_k; x_1,\ldots,x_k]$$

$$= F_C^{(k)}[\tfrac{1}{2}a, \tfrac{1}{2}a+\tfrac{1}{2}; c_1,\ldots,c_k; 4x_1,\ldots,4x_k] \tag{1.12.9}$$

where $F_A^{(n)}$ and $F_C^{(n)}$ are Lauricella functions of n-variables defined by (1.10.1) and (1.10.3) respectively.

1.13 MULTIPLE HYPERGEOMETRIC SERIES ${}^{(k)}_{(1)}E_D^{(n)}$ and ${}^{(k)}_{(1)}E_C^{(n)}$

Exton [27,p.89(3.4.1)] considered a multiple hypergeometric function which follow as generalization of certain of the quadruple functions.

This function is defined as follows :

$$ {}^{(k)}_{(1)}E^{(n)}_{D}\,[a, b_1,\ldots,b_n; c, c'; x_1,\ldots\ldots,x_n] $$

$$ = \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}(b_1)_{m_1}\ldots(b_n)_{m_n}}{(c)_{m_1+\ldots+m_k}(c')_{m_{k+1}+\ldots+m_n}} \frac{x_1^{m_1}}{m_1!}\ldots\ldots\frac{x_n^{m_n}}{m_n!}. \qquad (1.13.1) $$

Prompted by the above work of Exton, Chandel [13] also defined and studied the function.

$$ {}^{(k)}_{(1)}E^{(n)}_{C}\,[a, a', b; c_1,\ldots, c_n; x_1,\ldots\ldots,x_n] $$

$$ = \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_k}(a')_{m_{k+1}+\ldots+m_n}(b_n)_{m_1+\ldots+m_n}}{(c_1)_{m_1}\ldots\ldots\ldots\ldots(c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}\ldots\ldots\frac{x_n^{m_n}}{m_n!}. \qquad (1.13.2) $$

The region of convergence of the functions (1.13.1) and (1.13.1) are given in [27,p.91-92].

We note the following special cases (cf.[27,p.90-91]) :

(i) $ {}^{(0)}_{(1)}E^{(n)}_{C}\,[a, b; c_1,\ldots,c_n; x_1,\ldots,x_n] $

$$ = F_C^{(n)}\,[a, b; c_1,\ldots,c_n; x_1,\ldots\ldots,x_n]. \qquad (1.13.3) $$

(ii) $ {}^{(0)}_{(1)}E^{(n)}_{D}\,[a, b_1,\ldots,b_n; c; x_1, \ldots,x_n] $

$$ = F_D^{(n)}\,[a, b_1,\ldots,b_n; c; x_1,\ldots,x_n]. \qquad (1.13.4) $$

where $F_C^{(n)}$ and $F_D^{(n)}$ are Lauricella functions defined by (1.10.3) and (1.10.4) respectively.

(iii) $ {}^{(1)}_{(1)}E^{(2)}_{C}\,[a, a', b; c_1, c_2; x, y] = F_2\,[b,a,a'; c_1,c_2; x, y]. \qquad (1.13.5) $

(iv) $ {}^{(1)}_{(1)}E^{(2)}_{D}\,[a, b_1, b_2; c_1, c_2; x, y] = F_2\,[a, b_1, b_2; c_1, c_2; x, y], \qquad (1.13.6) $

where F_2 is Appell function of two variables defined by (1.4.2).

(v) $\,^{(1)}_{(1)}E^{(n)}_{C}$ [a, a', b; c_1; c_2, c_3; x, y, z]

$$= F_E\,[b, b, b, a, a', a'; c_1, c_2, c_3; x, y, z]\,. \qquad (1.13.7)$$

(vi) $\,^{(1)}_{(1)}E^{(3)}_{D}$ [a, b_1, b_2, b_3; c, c'; x, y, z]

$$= F_G\,[a, a, a, b_1, b_2, b_3; c, c'; x, y, z]\,, \qquad (1.13.8)$$

where F_E and F_G are Lauricella's triple hypergeometric series defined as follows [93, p. 42(1) and (3)]:

$$F_E\,[a, a, a, b, d, d; c_1, c_2, c_3; x, y, z]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n+p}\,(b)_m\,(d)_{n+p}}{(c_1)_m\,(c_2)_n\,(c_3)_p}\;\frac{x^m}{m!}\;\frac{y^n}{n!}\;\frac{z^p}{p!} \qquad (1.13.9)$$

$$|x|<r,\ |y|<s,\ |z|<t,\ r+(\sqrt{s}+\sqrt{t})^2 = 1.$$

$$F_G\,[a, a, a, b_1, b_2, b_3; c_1, c_2, c_3; x, y, z]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n+p}\,(b)_m\,(b_2)_n\,(b_3)_p}{(c_1)_m\,(c_2)_{n+p}}\;\frac{x^m}{m!}\;\frac{y^n}{n!}\;\frac{z^p}{p!} \qquad (1.13.10)$$

$$|x|<r,\ |y|<s,\ |z|<t,\ r+s=1=r+t.$$

(vii) $\,^{(3)}_{(1)}E^{(4)}_{C}$ [a, a', b; c_1, c_2, c_3, c_4, x, y, z, w]

$$= K_2\,[b, b, b, b, a, a, a, a'; c_1, c_2, c_3, c_4; x, y, z, w]. \qquad (1.13.11)$$

(viii) $\,^{(2)}_{(1)}E^{(4)}_{C}$ [a, a', b; c_1, c_2, c_3, c_4; x, y, z, w]

$$= K_5\,[b, b, b, b; a, a, a', a'; c_1, c_2, c_3, c_4; x, y, z, w]. \qquad (1.13.12)$$

(ix) $\,^{(3)}_{(1)}E^{(4)}_{D}$ [a, b_1, b_2, b_3, b_4; c, c'; x, y, z, w]

$$= K_{11}\,[a, a, a, a; b_1, b_2, b_3, b_4; c, c, c, c'; x, y, z, w]. \qquad (1.13.13)$$

(x) $\,^{(2)}_{(1)}E^{(4)}_{D}$ [a, b_1, b_2, b_3, b_4; c, c'; x, y, z, w]

$$= K_{12}\,[a, a, a, a; b_1, b_2, b_3, b_4; c, c, c', c'; x, y, z, w]. \qquad (1.13.14)$$

where K_2, K_5, K_{11} and K_{12} are Exton's quadruple hypergeometric functions (cf. [27,p.77-79]).

1.14 ORTHOGONAL POLYNOMIALS

Orthogonal polynomials are of great importance in mathematical physics, approximation theory, the theory of mechanical quadratures, etc. This class contains many special functions commonly encountered in the applications, e.g. Legendre, Hermite, Gegenbauer, Jacobi and Rice polynomials. Orthogonal polynomials are treated in many excellant books such as Rainville [73], Lebedev [57] and Prudnikov et al. (see [70] and [71]). Some of the orthogonal polynomials and their connections with hypergeometric functions used in our work are given below.

Hermite Polynomials

Hermite polynomials are defined by means of generating relation

$$\exp\,(2xt - x^2) = \sum_{n=0}^{\infty} H_n(x)\,\frac{t^n}{n!}, \qquad (1.14.1)$$

valid for all finite x and t.

It follow from (1.14.1) that

$$H_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k \, n! \, (2x)^{n-2k}}{k! \quad (n-2k)!}, \tag{1.14.2}$$

or equivalently

$$H_n(x) = (2x)^n \, {}_2F_0 \, [-n/2, \tfrac{1}{2} - n/2; —; -(1/x^2)] \, . \tag{1.14.3}$$

Associated Laguerre Polynomials

The associated Laguerre polynomials $L_n^{(\alpha)}(x)$ are defined by means of the generating relation

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x) \, t^n = (1-t)^{-(\alpha+1)} \exp(-xt), \tag{1.14.4}$$

where

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!} \, . \tag{1.14.5}$$

The hypergeometric form of the Laguerre polynomials given in Rainvile [73, p. 200 (1)] is

$$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!} \, {}_1F_1 \, [-n; 1+\alpha; x] \tag{1.14.6}$$

$$(\operatorname{Re}(\alpha) > -1),$$

where the factor $\dfrac{(1+\alpha)_n}{n!}$ is inserted for the sake of convenience only.

For $\alpha = 0$, each of the above equations yield expressions of simple Laguerre polynomials $L_n(x)$.

Jacobi Polynomials

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are defined by the generating relation

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)\, t^n = [1+½(x+1)t]^{\alpha}\ [1+½\,(x-1)]^{\beta}\,, \tag{1.14.7}$$

$$(\mathrm{Re}\,(\alpha) > -1,\ \mathrm{Re}\,(\beta) > -1).$$

The Jacobi polynomials have a number of finite series representations, (see [73,p. 255] and [94, p. 91]), one of them is given below [73, p.255(4)]

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \frac{(1+\alpha)_n\,(1+\alpha+\beta)_{n+k}}{k!\,(n-k)!\,(1+\alpha)_k\,(1+\alpha+\beta)_n} \left(\frac{x-1}{2}\right)^k. \tag{1.14.8}$$

For $\beta=\alpha$, the Jacobi polynomials in (1.14.8) reduces to Gegenbauer polynomials $C_n^{(\alpha)}(x)$, given by

$$C_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!}\ {}_2F_1\left[\begin{matrix}-n,\ 1+2\alpha+n; & \dfrac{1-x}{2}\\ \alpha+1; & \end{matrix}\right] \tag{1.14.9}$$

When, we take $\alpha = \beta = 0$, in equation (1.14.8), we get the Legendre polynomials $P_n(x)$ given in [73, p. 166 (2)] by

$$P_n(x) = {}_2F_1\left[-n,\ n+1;\ 1;\ \frac{1-x}{2}\right] \tag{1.14.10}$$

The Laguerre polynomials $L_n^{(\alpha)}(x)$ defined by (1.14.5) and the generalized Bessel polynomials $y_n(a, b; x)$ are, in fact, limiting cases of the Jacobi polynomials (see [94, p. 131(1)] and [1,p.411(2)]).

$$L_n^{(\alpha)}(x) = \lim_{|\beta|\to 0} P_n^{(\alpha,\beta)}\left(1 - \frac{2x}{\beta}\right), \tag{1.14.11}$$

$$y_n(a, b; x) = \lim_{\beta \to 0} \frac{\Gamma(n+1)\,\Gamma(\beta)}{\Gamma(\beta+n)} \, P_n^{(\beta-1,\, \alpha-\beta-1)}\left(1 - \frac{2x\beta}{b}\right), \qquad (1.14.12)$$

where the hypergeometric form of the generlaized Bessel polynomials $y_n(a, b; x)$ given in Krall and Frink work [53] is

$$y_n(a, b; x) = {}_2F_0\left[-n, a-1+n; —; \frac{-x}{b}\right]. \qquad (1.14.13)$$

<u>Generalized Rice Polynomials</u>

Investigation of Rice [75], were continued by Khandeker [51] who in 1964 defined the generalized Rice polynomials

$$H_n^{(\alpha,\beta)}[\xi, p, v] = \binom{\alpha+n}{n} {}_3F_2[-n, \alpha+\beta+n+1, \xi; \alpha+1, p; v] \qquad (1.14.14)$$

$$(\mathrm{Re}(\alpha) > -1, \mathrm{Re}(\beta) > -1).$$

In 1972, Khan (see [49] and [50]) introduced a generlaization of Rice polynomials defined by (1.14.14) in the form

$$f_n^{(\alpha,\beta)}[(a_p), (b_q); x] = \frac{(1+\alpha)_n}{n} \, {}_{p+2}F_{q+1}\left[\begin{array}{ll} -n, 1+\alpha+\beta+n, (a_p); & \\ 1+\alpha, \quad (b_q); & \end{array} x\right]. \qquad (1.14.15)$$

Clearly we note that

$$f_n^{(\alpha,\beta)}[\xi, p, v] = H_n^{(\alpha,\beta)}[\xi, p, v] \qquad (1.14.16)$$

and

$$P_n^{(\alpha,\beta)}(x) = H_n^{(\alpha,\beta)}[\xi, \xi, (1-x)/2]. \qquad (1.14.17)$$

Brafman's Polynomials

In 1957, Brafman studies the generalized hypergeometric function [9,p.180-187]

$$B_n^m\,[(a);\,(b);\,x] = {}_{m+A}F_B\left[\begin{matrix}\Delta\,(m;\,-n),\,(a);\\ (b);\end{matrix}\;x\right] \tag{1.14.18}$$

where m is positive integer, the parameters a_j (j=1,2,...,A) and b_j (j=1, 2,..,B) are independent of n and $\Delta(m;\,\lambda)$ abbreviates the array of m parameters $(\lambda+j-1)/m$, $j = 1, 2, \ldots, m$. Special cases of Brafman's polynomials (1.14.18) are, for example, the Gegenbauer polynomials since

$$C_n^{(\alpha)}(x) = \frac{(2\alpha)_n}{n!}\, x^n\, B_n^2\,[\,—;\,\alpha+\tfrac{1}{2};\,(x^2-1)/x^2\,]\,, \tag{1.14.19}$$

the Gold-Hopper polynomials defined as follows [94, p. 76]

$$g_n^m\,(x,\,h) = x^n\,{}_mF_0\,[\Delta(m;\,-n);\,—;\,h\,(-m/x)^m\,], \tag{1.14.20}$$

and the classical Hermite polynomials given in (1.14.2).

Bedient's Polynomials

In 1959, Bedient in his study of some polynomials associated with Appell's function F_2 introduced the polynomials $R_n(\alpha,\beta;x)$ [7,p. 15 (2.5)] by

$$R_n\,(\alpha,\beta;x) = \frac{(\alpha)_n\,(2x)^n}{n!}\,{}_3F_2\,[-n/2,\,\tfrac{1}{2}-n/2;\,\beta-\alpha;\,\beta,\,1-\alpha-n;\,1/x^2\,]. \tag{1.14.21}$$

1.15 GENERATING FUNCTIONS

A generating function may be used to define a set of functions, to determine a differential recurrence relation or a pure recurrence relation, to evaluate certain integrals, etc. We define a generating function for a set of functions $\{f_n(x)\}$ as follows [94, p.78-82] :

Definition : Let G(x,t) be a function that can be expanded in powers of t such that

$$G(x,t) = \sum_{n=0}^{\infty} C_n f_n(x)\, t^n, \tag{1.15.1}$$

where $\{C_n\}$ is a function of n, independent of x and t. Then G (x,t) is called a generating function of the set $\{f_n(x)\}$. By the above definition a set of functions may have more then one generating function.

However, if

$$G(x,t) = \sum_{n=0}^{\infty} h_n(x)\, t^n, \tag{1.15.2}$$

then G(x,t) is the unique generator for the set of functions $\{h_n(x)\}$ as the coefficient set. If the set of function $\{f_n(x)\}$ is also defined for negative integers n=0, ± 1, ± 2,..., the definition (1.15.1) may be extended in terms of the Laurent series expansion

$$G(x,t) = \sum_{n=-\infty}^{\infty} C_n f_n(x)\, t^n, \tag{1.15.3}$$

where $\{C_n\}$ is independent of x and t. The above definition of generating function, used earlier by Rainville [73, p. 129] and Mc Bride [62, p.1], may be extended to include generating functions of several variables.

Definition Let $G(x_1,...,x_k; t)$ be a function of (k+1) variables, which has a formal expansion in power of t such that

$$G(x_1,....,x_k; t) = \sum_{n=0}^{\infty} C_n f_n(x_1,...,x_k)\, t^n, \tag{1.15.4}$$

where the sequence C_n is independent of the variables $x_1,....,x_k$ and t. Then we shall say that $G(x_1,...,x_k; t)$ is a multivariable generating function for the set $\{f_n(x_1,...,x_k)\}_{n=0}^{\infty}$, corresponding to non-zero coefficients $\{C_n\}$.

Bilinear and Bilateral Generating Functions

A multivariable generating function $G(x_1,.....,x_k; t)$ given by (1.15.4) is said to be a multilateral generating function if

$$f_n(x_1,....,x_k) = g_{1\alpha_1(n)}(x_1).........g_{k\alpha_k(n)}(x_n), \tag{1.15.5}$$

where $\alpha_j(n)$, j=1,2,...,k, are functions of n which are not necessarily equal. Moreover, if the functions occuring on the right hand side of (1.15.5) are all equal, the equation (1.15.4) will be called a multilinear generating function.

In particular, if

$$G(x, y; t) = \sum_{n=0}^{\infty} C_n\, f_n(x)\, g_n(y)\, t^n, \tag{1.15.6}$$

and the sets $\{f_n(x)\}_{n=0}^{\infty}$ and $\{g_n(y)\}_{n=0}^{\infty}$ are different the function $G(x,y;t)$ is called bilateral generating function for the set $\{f_n(x)\}$ or $\{g_n(x)\}$.

If $\{f_n(x)\}_{n=0}^{\infty}$ and $\{g_n(y)\}_{n=0}^{\infty}$ are same set of functions then in that case, we shall say that $G(x,y;t)$ is a bilinear generating function for the set $\{f_n(x)\}$.

CHAPTER-II

A CERTAIN CLASS OF GENERATING FUNCTIONS INVOLVING BILATERAL SERIES

2.1 INTRODUCTION

Let

$$F_n^m(x) = \frac{1}{m!\ n!}\ {}_1F_1\ [\ -n;\ m+1;\ x\ ] \tag{2.1.1}$$

$$= \frac{1}{(m+n)!}\ L_n^{(\alpha)}(x)\ , \tag{2.1.2}$$

where $L_n^{(\alpha)}(x)$ denotes the classical Laguerre polynomials [97] defined by (1.14.5). An interesting (partly bilateral and partly unilateral) generating function for $F_n^{(m)}(x)$, due to Exton [31,p.147(3)], is recalled here in the following (modified) from (cf. [68] and [69]) :

$$\exp(s+t-xt/s) = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} s^m\ t^n\ F_n^m\ (x) \tag{2.1.3}$$

or equivalently

$$\exp(s+t-xt/s) = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} {}_1F_1\ [-n,\ m+1;x]\ \frac{s^m}{m!}\ \frac{t^n}{n!} \tag{2.1.4}$$

where, and in what follow

$$m^* = \max(0,\ -m) \qquad \left(m\in\mathbb{Z} = \{0,\ \pm1,\ \pm2,\ \ldots.\}\right), \tag{2.1.5}$$

and

$$F_n^m\ (x) = \frac{L_n^{(m)}\ (x)}{(m\ +\ n)\ !} = \frac{1}{n!} \sum_{r=m^*}^{n} \frac{(-n)_r\ x^r}{(m+r)!\ r!}\ ,\ \text{if } n\geq m^*,$$

$$= 0\ ,\ \text{if } 0\leq n<m^*\ (\text{i.e. if } m+n\leq 0\ \leq n),$$

so that all factorials of negative integers accuring in this definition have a meaning. Exton's generating function (2.1.3) has since been extended by a number of workers including (for example) Pathan and Yasmeen ([68] and [69]), Kamarujjama et al. [46], Srivastava et al. [96], Gupta et al. [39] and Goyal and Rajni [36].

This chapter amis at presenting a general theorem on partly bilateral and partly unilateral generating relations. Section 2.2 deals with the motivation to establish a new class of (partly bilateral and partly unilateral) generating functions involving Shively [78] generalized hypergeometric function. In section 2.3 many special cases involving the product of three polynomials of Bateman, Bessel, Hermite, Rice, Jacobi and Gegenbauer are obtained. Section 2.4 aims at presenting a general theorem on partly bilateral and partly unilateral generating functions invloving multiple series with essentially arbitrary coefficients. By appropriate specializing these coefficients, a number of (known and new) results are shown to follow as applications of the theorem. Also, of interest are erroneous results due to Kamarujjama et al. [46, p. 362, Equations (3.1) and (3.2)] which are corrected here.

2.2 NEW GENERATING RELATION INVLOVING SHIVELY'S FUNCTION

Shively [78] introduced an interesting generalized hypergeometric function, defined by

$$S_h^{\alpha}(x) = \frac{(\alpha+h)_h}{h!}\, {}_{p+1}F_{q+1}\left[\begin{matrix} -h,\ a_1,\dots\dots,a_p; \\ \alpha+h,\ b_1,\dots\dots,b_q; \end{matrix}\ x\right] = \frac{(\alpha+h)_h}{h!} \sum_{k=0}^{h} \frac{A_k\, x^k}{(\alpha+h)_k} \quad (2.2.1)$$

where $A_k = \dfrac{(-h)_k \ (a_1)_k \ \cdots\cdots\cdots \ (a_p)_k}{k! \ \ (b_1)_k \ \cdots\cdots\cdots (b_q)_k}$.

Here p and q are positive integers or zero and we assume that the variables x, the numerator parameters $a_1, \ldots\ldots, a_p$ and the denominator parameters $b_1, \ldots\ldots b_q$ take on complex values, provided that the denominator parameters $\alpha+h$ and $b_j \neq 0, -1, -2, \ldots\ldots, (j=1, 2\ldots\ldots q)$.

Generating relations of the type (2.1.3) for many classes of polynomials are generally not known, suggest that a set of generating relations also exist which may be obtained in similar manner as in (2.1.3). In an attempt to obtain such new relation by using shively's polynomials (2.2.1), we have found the following new generating relation which is partly bilateral and partly unilateral.

$$\sum_{m,n,k=0}^{\infty} \frac{A_m B_n C_k \, y^m \, z^n \, (-xz/y)^k}{(\alpha+p)_m \, (\beta+q)_n \, (\delta+r)_k} = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{y^m \, z^n}{(\alpha+p)_m \, (\beta+q)_n}$$

$$\sum_{k=0}^{n} \frac{(1-\beta-q-n)_k \, C_k \, A_{m+k} \, B_{n-k} \, x^k}{(\alpha+p+m)_k \, (\delta+r)_k} \qquad (2.2.2)$$

<u>Derivation of the main generating function (2.2.2)</u>

Let V denotes the left-hand side of the relation (2.2.2). Then by replacing m and n respectively by i and j, we readily observe that

$$V = \sum_{i=0}^{\infty} \frac{A_i \, y^{i-k}}{(\alpha+p)_i} \sum_{j=0}^{\infty} \frac{B_j \, z^{j+k}}{(\beta+q)_j} \sum_{k=0}^{\infty} \frac{C_k \, (-x)^k}{(\delta+r)_k}$$

Let i-k=m and j+k=n, then after rearrangement, justified by the absolute convergence of the above series and use of (1.1.11) and (1.1.13) it follows

$$V=\sum_{m=-\infty}^{\infty} \frac{A_{m+k}\, y^m}{(\alpha+p)_{m+n}} \sum_{n=m^*}^{\infty} \frac{B_{n-k}\, Z^n}{(\beta+q)_{n-k}} \sum_{k=0}^{\infty} \frac{C_k\, (-x)^k}{(\delta+r)_k}$$

which is equivalent to

$$V=\sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{y^m\, z^n}{(\alpha+p)_m\, (\beta+q)_n} \sum_{k=0}^{n} \frac{(1-\beta-q-n)_k\, C_k\, A_{m+k}\, B_{n-k}\, X^k}{(\alpha+p+m)_k\, (\delta+r)_k}$$

This completes the proof of (2.2.2).

2.3. SPECIAL CASES

Equation (2.2.2) gives many generating functions for well known polynomials. We present some special cases here.

$$\text{Set } A_m = \frac{(-p)_m\, (a_i)_m}{m!\, (b_j)_m}, \quad B_n = \frac{(-q)_n\, (c_l)_n}{n!\, (d_s)_n}, \quad C_k = \frac{(-r)_k\, (e_u)_k}{k!\, (f_v)_k}, \tag{2.3.1}$$

in (2.2.2) to get

$${}_{i+1}F_{j+1}\left[\begin{matrix}-p,\ (a_i); \\ \alpha+p, (b_j);\end{matrix}\ y\right] {}_{l+1}F_{s+1}\left[\begin{matrix}-q,\ (c_l); \\ \beta+q,\ (d_s);\end{matrix}\ Z\right] {}_{u+1}F_{v+1}\left[\begin{matrix}-r,\ (e_u); \\ \delta+r,\ (f_v);\end{matrix}\ -xz/y\right]$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{(-p)_m\, (-q)_n\, (a_i)_m\, (c_l)_n\, y^m\, z^n}{(\alpha+p)_m\, (\beta+q)_n\, (b_j)_m\, (d_s)_n\, m!\, n!}$$

$${}_{i+s+u+4}F_{j+l+v+4}\left[\begin{matrix}-p+m,\ -r,\ (a_i)+m,\ 1-(d_s)-n,\ e_u,\ 1-\beta-q-n, -n\ ; \\ 1+q-n,\ (b_j)+m, 1-(c_l)-n,\ f_v,\ \delta+r,\ \alpha+p+m,\ m+1\ ;\end{matrix}\ (-1)^{l-s}\ x\right] \tag{2.3.2}$$

For $\alpha = -2p$, $\beta = -2q$ and $\delta = -2r$, (2.3.2) reduces to a known result of Pathan and Yasmeen [69, p.3 (1.5)].

If in (2.3.2), we set $i=j=l=s=u=v=0$, $\alpha = -2p$, $\beta = -2q$ and $\delta = -2r$, then it reduces to a known result of Exton [31, p. 147 (3)].

On setting $A_m = \dfrac{(\alpha+p)_m(\alpha)_m}{m!}$, $B_n = (\beta+q)_n$,$C_k = \dfrac{(\delta+r)_k\ (\beta)_k}{k!}$, $z=1$

in (2.2.2), we get

$$(1-y)^{-\alpha}\left(1+\frac{x}{y}\right)^{-\beta} = \sum_{m=-\infty}^{\infty} \frac{\Gamma(\alpha+m)\quad y^m}{\Gamma(\alpha)\quad \Gamma(m+1)}\ {}_2F_1\left[\begin{matrix}\alpha+m,\ \beta; \\ m+1;\end{matrix}\ -x\right]$$

which is a known result [94, p.325 (9)].

For $\alpha = \alpha - 3/2p+1$, $\beta = \beta - 3/2q+1$, $\delta = \delta - 3/2r+1$, (2.3.2) gives

$$f_p^{\alpha}(y)\ f_q^{\beta}(z)\ f_r^{\delta}(-xz/y) = \binom{\alpha+\frac{1}{2}p}{p}\binom{\beta+\frac{1}{2}q}{q}\binom{\delta+\frac{1}{2}r}{r}$$

$$\sum_{m=-\infty}^{\infty}\ \sum_{n=m^*}^{\infty} \frac{(-p)_m\ (-q)_n\ (a_i)_m\ (c_l)_n \qquad y^m\ z^n}{(\alpha-\frac{1}{2}p+1)_m\ (\beta-\frac{1}{2}q+1)_n\ (b_j)_m\ (d_s)_n\ m!\ n!}$$

$${}_{i+s+u+4}F_{j+l+v+u}\left[\begin{matrix}-p+m,\ -r,\ (a_i)+m,\ 1-(d_s)-n,\ e_u,\ -\beta+\frac{1}{2}q-n,\ -n; \\ 1+q-n, (b_j)+m, 1-(c_l)-n, f_v,\ \delta-\frac{1}{2}r+1, \alpha-\frac{1}{2}p+m+1, m+1;\end{matrix}\ (-1)^{l-s}\, x\right] \tag{2.3.3}$$

where $f_n^{\alpha}(x) = \binom{\alpha+\frac{1}{2}n}{n}\ {}_{p+1}F_{q+1}\left[\begin{matrix}-n,\ a_1,\ldots\ldots,a_p; \\ \alpha-\frac{1}{2}n+1,\ b_1,\ldots\ldots,b_q;\end{matrix}\ x\right]$ is

Brown's generalised hypergeometric polynomials [10].

Putting $i=g+1$, $l=t+1$, $u=h+1$, $a_{g+1} = \alpha+g_1+2p$, $c_{t+1} = \beta+g_2+2q$, $e_{h+1} = \delta+g_3+2r$ $\alpha = \alpha - p+1$, $\beta = \beta - q+1$, $\delta = \delta - r+1$, in (2.3.2), we get

$$f_p^{(\alpha,g_1)}[(a_g);\ (b_j):y]\ f_q^{(\beta,g_2)}[(c_t);\ (d_s):z]\ f_r^{(\delta,g_3)}[(e_h);\ (f_v);\ -xz/y]$$

$$= \binom{\alpha+p}{p}\binom{\beta+q}{q}\binom{\delta+r}{r}\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{(-p)_m(-q)_n(\alpha+g_1+p+1)_m(a_g)_m(c_t)_n(\beta+g_2+q+1)_n\,y^m z^n}{(\alpha+1)_m(\beta+1)_n(b_j)_m(d_s)_n\,m!\,n!}$$

$$_{g+s+h+6}F_{j+t+v+5}\left[\begin{matrix}-p+m,\ -r,\ \alpha+g_1+p+m+1,\ \delta+g_3+r+1,\ (a_g)+m\ 1-(d_s)-n,\\ 1+q-n,\ -\beta-g_2-q-n,\ (b_j)+m,\ 1-(c_t)-n,\ f_v,\ \alpha+m+1,\end{matrix}\right.$$

$$\left.\begin{matrix}e_h,\ -\beta-n,\ -n;\\ \delta+1,\ m+1;\end{matrix}\ (-1)^{t-s+1}(x)\right] \tag{2.3.4}$$

where $f_n^{(\alpha,g)}\left[(a_p);\ (b_q):\ x\right]=\binom{\alpha+n}{n}\ _{p+2}F_{q+1}\left[\begin{matrix}-n,\ \alpha+g+n+1,\ a_1,\ldots,a_p;\\ \alpha+1,\ b_1,\ldots\ldots b_q;\end{matrix}\ x\right]$

is a special class of hypergeometric polynomials [94, p.177 (32)].

On taking $i=j=l=s=u=v=1$, $a_1=p+1$, $c_1=q+1$, $e_1=r+1$, $b_1=f_1=d_1=1$, $\alpha=1-p$, $\beta=1-q$, $\delta=1-r$, in (2.3.2), we obtain

$$Z_p(y)\ Z_q(z)\ Z_r\ (-xz/y)=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{(-p)_m\ (-q)_n\ (p+1)_m\ (q+1)_n\ y^m\ z^n}{((1+n))^2\ ((\ 1+m))^2\ m!\ n!}$$

$$_7F_7\left[\begin{matrix}-p+m,\ -r,\ p+m+1,\ r+1,\ -n,\ -n,\ -n;\\ 1+q-n,\ -q-n,\ 1,\ 1,\ m+1,\ m+1,\ m+1;\end{matrix}\ (-1)^{l-s}\ x\right] \tag{2.3.5}$$

where $Z_n(x)$ is Bateman's polynomials [73, p. 285(2)]

$$Z_n\ (x)\ =\ _2F_2\ [-n,\ n+1;\ 1,\ 1;\ x].$$

Setting $i=l=u=0$, $j=s=v=1$, $\alpha=-1-p/2$, $\beta=-1-q/2$, $\delta=-1-r/2$, $b_1=1+\alpha$, $d_1=1+\beta$, $f_1=1+\delta$, replacing y, z and x by $-x^2/4$, $-y^2/4$, and $-z^2x^2/2y^2$, respectively (2.3.3) gives a known result of Pathan and Yasmeen [69, p.7 (3.4)].

If in (2.3.2), we set j=s=v=0, i=l=u=2, $a_1=\alpha+p$, $a_2= -p+1/2$, $c_1=\beta+q$, $c_2= -q+½$, $e_1=\delta+r$, $e_2= -r+½$, and replace y, z, x, p, q and r by $-1/x^2$, $-1/y^2$, y^2/z^2x^2, ½p, ½q and ½r respectively, we get

$$H_p(x)\ H_q(y)\ H_r(z) = (2x)^p\ (2y)^q\ (2z)^r \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{(-1)^{m+n}\ (-p/2)_m\ (-q/2)_n}{m!} \frac{(-p/2+½)_m\ (-q/2+½)_n\ x^{-2m}\ y^{-2n}}{n!}$$

$$ {}_8F_6\left[\begin{matrix} -½p+m,\ -½r,\ \alpha+½p+m,\ -½p+m+½,\ \delta+½r,\ -½r+½,\ 1-\beta-½q-n,\ -n; \\ 1+½q-n,\ 1-\beta-½q-n,\ ½+q-n,\ \delta+½r,\ \alpha+m+½p,\ m+1; \end{matrix}\ \frac{y^2}{z^2x^2}\right] \qquad (2.3.6)$$

where H_n (x) is Hermite Polynomial [73, p.191 (108)]

For g=j=t=s=h=v=1, a_1=a, b_1=b, c_1=c, d_1=d, e_1=e. f_1=f , (2.3.4) gives

$$H_p^{(\alpha,g_1)}[a,\ b,\ y]\quad H_q^{(\beta,g_2)}[c,\ d,\ z]\quad H_r^{(\delta,\ g_3)}[e,f\ -xz/y] =$$

$$\frac{(1+\alpha)_p}{p!}\ \frac{(1+\beta)_q}{q!}\ \frac{(1+\delta)_r}{r!} \sum_{m=-\infty}^{\infty} \sum_{n=m}^{\infty} \frac{(-p)_m(-q)_n\ (\alpha+g_1+p+1)_m\ (\beta+g_2+q+1)_n\ (a)_m\ (c)_n y^m\ z^n}{(1+\alpha)_m\ (1+\beta)_n\ (b)_m\ (d)_n\ m!\ n!}$$

$$ {}_9F_8\left[\begin{matrix} -p+m,\ -r,\ \alpha+g_1+p+m+1,\ \delta+g_3+r+1,\ a+m,\ 1-d-n,\ e,\ -\beta-n,\ -n; \\ 1+q-n,\ -\beta-g_2-q-n,\ b+m,\ 1-c-n,\ f,\ \alpha+m+1,\ \delta+1,\ m+1; \end{matrix}\ -x\right] \qquad (2.3.7)$$

where $H_n^{(\alpha,g)}$ [a, b, x] is generalized Rice polynomials [51, p. 158 (2.3)].

Further on taking $g_1=g_1-p$, $g_2=g_2-q$, $g_3=g_3-r$, (2.3.7) yields a known generating relation [69, p.6 (3.3)].

In (2.3.4), setting g=j=t=s=h=v=o, replacing y,z and x by

$\left(\frac{1-y}{2}\right)$, $\left(\frac{1-z}{2}\right)$ and $\left[\frac{-(1-y)\ (1-x)}{2\ (1-z)}\right]$ respectively, we get the following generating relation involving a product of Jacobi polynomials [73, p.254 (1)]

$$P_p^{(\alpha,g_1)}(y)\ P_q^{(\beta,g_2)}(z)\ P_r^{(\delta,g_3)}(x) = \frac{(1+\alpha)_p\ (1+\beta)_q\ (1+\delta)_r}{p!\ q!\ r!}$$

$$\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty} \frac{(-p)_m\ (-q)_n\ (\alpha+q_1+1)_m\ (\beta+g_2+q+1)_n\ ((1-y)/2)^m\ ((1-z)/2)^n}{(1+\alpha)_m\ (1+\beta)_n\ m!\ n!}$$

$${}_6F_5\left[\begin{matrix} -p+m,\ \alpha+g_1+p+m+1,\ -r,\ \delta+g_3+r+1,\ -\beta-n,\ -n; \\ 1+q-n,\ -\beta-g_2-q-n,\ \alpha+m+1,\ \delta+1,\ m+1; \end{matrix}\ ((1-y)(1-x))/(2(1-z))\right] \tag{2.3.8}$$

which further on replacing y, z and x by (1-2y) (1-2z) and (1+2xz/y) respectively and taking $g_1=g_1-p$, $g_2=g_2-q$, g_3-g_3-r, yields a known generating relation [69, p. 6 (3.2)].

On taking g=j=t=s=h=v=0, $\alpha=\alpha-½$, $\beta=\beta-½$, $\delta=\delta-½$, $g_1=-½+\alpha$, $g_2=-½+\beta$, $g_3=-½+\delta$, replacing y, z and x by (1-y)/2, (1-z)/2, and -(1-y)(1-x)/2(1-z) respectively, in (2.3.8) and using [94,p.35(35)], we get the following generating relation involving a product of three Gegenbaur polynomials [73, p. 279 (15)].

$$C_p^{\alpha}(y)\ C_q^{\beta}(z)\ C_r^{\delta}(x) = \frac{(2\alpha)_p\ (2\beta)_q\ (2\delta)_r}{p!\ q!\ r!} \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty} \frac{(-p)_m\ (-q)_n\ (2\alpha+p)_m}{(\alpha+½)_m\ (\beta+½)_n}$$

$$\frac{(2\beta+q)_n}{m!\ n!}\left(\frac{1-y}{2}\right)^m\left(\frac{1-z}{2}\right)^n$$

$${}_6F_5\left[\begin{matrix} -p+m,\ 2\alpha+p+m,\ -r,\ 2\delta+r,\ -\beta-n+½,\ -n; \\ 1+q-n,\ -2\beta-q-n-1,\ \alpha+m+1,\ \delta+½,\ m+1; \end{matrix}\ \frac{(1-y)\ (1-x)}{2(1-z)}\right], \tag{2.3.9}$$

which further on replacing y, z and x by (1-2y), (1-2z) and (1+2xz/y) respectively, yields

$$C_p^{\alpha}(1-2y)\ C_q^{\beta}(1-2z)\ C_r^{\delta}(1+2xz/y) = \frac{(2\alpha)_p\ (2\beta)_q\ (2\delta)_r}{p!\ q!\ r!}$$

$$\sum_{m=-\infty}^{\infty}\ \sum_{n=m^*}^{\infty} \frac{(-p)_m\ (-q)_n\ (2\alpha+p)_m\ (2\beta+q)_n}{(\alpha+\frac{1}{2})_n\ \ (\beta+\frac{1}{2})_m\ \ m!\ \ n!}\ y^m\ z^n$$

$${}_6F_5\left[\begin{matrix}-p+m,\ 2\alpha+p+m,\ -r,\ 2\delta+r,\ -\beta-n+\frac{1}{2},\ -n\ ; \\ 1+q-n,\ -2\beta-q-n+1,\ \alpha+m+1,\ \delta+\frac{1}{2},\ m+1;\end{matrix}\ -x\right]. \tag{2.3.10}$$

2.4 GENERAL THEOREM ON PARTLY BILATREAL AND PARTLY UNILATERAL GENERATING RELATIONS

Motivated by the results of the previous section, the work of Pathan and Yasmeen ([68]) and [69]) and largely by the work of Kamarujjama et al. [46] in which the generating function in (2.1.4) was extended to hold true for the product of three Hubbell-Srivastava functions $\omega^{v}_{N}(x)$ defined by [43,p.351, Equation (3.1)]

$$\omega^{v}_{N}(x) = (v)_N \sum_{k=0}^{\infty} \frac{\Omega_k\ x^{N-2k}}{(1-v-N)_k}\ , \tag{2.4.1}$$

where $\{\Omega_n\}_{n=0}^{\infty}$ is a suitably bounded sequence of complex numbers, and the parameters v and **N** are unrestricted in general, we aim here at presenting a general theorem on partly bilateral and partly unilateral generating functions involving multiple series with essentially arbitrary coefficients. We also show how a number of (known or new) results can be deduced from the theorem by appropriately specializing these coefficients.

For covenience, a few conventions and notations are introduced here:

1. Boldface letters denote vectors of dimension r; for instance, we have

$$\mathbf{m} = (m_1,\ldots,m_r),\ \mathbf{n} = (n_1,\ldots,n_r),\ \text{and}\ \mathbf{k} = (k_1,\ldots,k_r).$$

2. The symbol $\{\Omega(m, n, k)\}$ denotes a triple sequence and the symbol $\{\Omega(\mathbf{m}, \mathbf{n}, \mathbf{k})\}$ denotes a multiple (3r-dimensional) sequence :

$$\{\Omega(m_1,\ldots,m_r;\ n_1,\ldots,n_r;\ k_1,\ldots,k_r)\}.$$

Sufficient conditions to ensure absolute convergence are understood to hold true, but each of these sequences is otherwise arbitrary.

3. Quite frequently, multiple series are written in simplified notation. Thus for, p, q $\in \mathbb{Z}$,

$$\sum_{\mathbf{m}=p}^{q} \text{ means } \sum_{m_1=p}^{q}\cdots\cdots\sum_{m_r=p}^{q} \quad ,$$

$$\sum_{\mathbf{m},\mathbf{n}=p}^{q} \text{ means } \sum_{m_1=p}^{q}\cdots\cdots\sum_{m_r=p}^{q} \cdot \sum_{n_1=p}^{q}\cdots\cdots\sum_{n_r=p}^{q} \ ,$$

$$\sum_{\mathbf{m},\mathbf{n},\mathbf{k}=p}^{q} \text{ means } \sum_{m_1=p}^{q}\cdots\sum_{m_r=p}^{q} \cdot \sum_{n_1=p}^{q}\cdots\sum_{n_r=p}^{q} \cdot \sum_{k_1=p}^{q}\cdots\sum_{k_r=p}^{q},$$

with the usual meaning when p or q (or both p and q) are replaced by r-dimensional vectors with integer elements, so that (for example)

$$\sum_{\mathbf{k}=\mathbf{p}}^{\mathbf{q}} \text{ means } \sum_{k_1=p_1}^{q_1}\cdots\cdots\sum_{k_r=p_r}^{q_r}$$

Our main results on generating functions involving bilateral series are given by the following

__Theorem__. Let $\{\Omega(m,n,k)\}$ be a suitably bounded triple sequence of complex numbers. Also let m^* be defined by (2.1.5). Then

$$\sum_{m,n,k=0}^{\infty} \Omega(m,n,k) \frac{y^m}{m!} \frac{z^n}{n!} \frac{(-xz/y)^k}{k!}$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{y^m \quad z^n}{m! \; n!} \sum_{k=0}^{n} \binom{n}{k} \Omega(m+k,\ n-k,\ k) \frac{(-x)^k}{(m+1)_k}, \quad (2.4.2)$$

provided that each member of (2.4.2) exists.

More generally, for a suitably bounded multiple (3r-dimensional) sequence $\{\Omega(\mathbf{m},\mathbf{n},\mathbf{k})\}$ of complex number, if

$m^* = (m^*_1,\ldots,m^*_r)$ with $m^*_j = \max(0,-m_j)$ $(m_j \in \mathbb{Z};\ j=1,\ldots,r)$, then (2.4.3)

$$\sum_{\mathbf{m},\mathbf{n},\mathbf{k}=0}^{\infty} \Omega(\mathbf{m},\mathbf{n},\mathbf{k}) \prod_{j=0}^{r} \left\{ \frac{y_j^{m_j}}{m_j!} \frac{z_j^{n_j}}{n_j!} \frac{(-x_j z_j/y_j)^{k_j}}{k_j!} \right\}$$

$$= \sum_{\mathbf{m}=-\infty}^{\infty} \sum_{\mathbf{n}=\mathbf{m}^*}^{\infty} \prod_{j=1}^{r} \left\{ \frac{y_j^{m_j}}{m_j!} \frac{z_j^{n_j}}{n_j!} \right\} \sum_{\mathbf{k}=0}^{\mathbf{n}} \Omega(\mathbf{m}+\mathbf{k},\ \mathbf{n}-\mathbf{k},\ \mathbf{k})$$

$$\prod_{j=1}^{r} \left\{ \binom{n_j}{k_j} \frac{(-x_j)^{k_j}}{(m_j+1)_{k_j}} \right\}, \quad (2.4.4)$$

provided that each member of (2.4.4) exists.

__Proof of theorem__ : Denote, for convenience, the first member of the assertion (2.4.2) by $S(x,y,z)$. Then it is easily seen that

$$S(x,y,z) = \sum_{m,n,k=0}^{\infty} \Omega\ (m,\ n,\ k) \frac{y^{m-k}}{m!} \frac{z^{n+k}}{n!} \frac{(-x)^k}{k!}.$$

Upon replacing the summation indices m and n by m+k and n–k, respectively, if we rearrange the resulting triple series (which can be justified by absolute covergence of the series involved), we are led finally to the generating function (2.4.2).

The derivation of the (multidimentional) assertion (2.4.4) runs paralled to that of (2.4.2).

2.5 APPLICATIONS OF THE THEOREM

First of all, in its special case when

$$\Omega\ (m,\ n,\ k) \equiv 1, \tag{2.5.1}$$

the assertion (2.4.2) would obviously correspond to the generating functions (2.1.3) and (2.1.4). Secondly, upon setting

$$\Omega(m,n,k) = (\lambda)_L\ (\mu)_M (\nu)_N\ \frac{m!\ \Omega'_m}{(1-\lambda-L)_m}\ \frac{n!\ \Omega_n''}{(1-\mu-M)_n}\ \frac{k!\ \Omega_k'''}{(1-\nu-N)_k} \tag{2.5.2}$$

in terms of the sequences $\{\Omega_n^{(j)}\}$ $(j = 1,2,3)$ and the (essentially unrestricted) parameters λ, μ, ν and L, M, N, if we make the following variable changes :

$$x \to -x^{-2},\ y \to y^{-2},\ \text{and}\ z \to z^{-2},$$

and apply the definition (2.4.1), we shall obtain a partly bilateral and partly unilateral generating function for the product of three Hubbell-Srivastava functions in the form :

$$\omega^{\lambda}_L(y)\ \omega^{\mu}_M\ (z)\ \ \omega^{\nu}_N\left(-\frac{xz}{y}\right) = (\lambda)_L\ (\mu)_M\ (\nu)_N$$

$$\cdot \sum_{m=-\infty}^{\infty}\ \sum_{n=m^*}^{\infty} \frac{y^{L-N-2m}\ z^{M+N-2n}}{(1-\lambda-L)_m\ (1-\mu-M)_n}\ \sum_{k=0}^{n} \frac{(-1)^k\ (\mu+M-n)_k}{(1-\lambda-L+m)_k\ (1-\nu-N)_k}$$

$$\cdot\ \Omega'_{m+k}\ \ \Omega''_{n-k}\ \Omega'''_k\ \ (-x)^{N-2k}\ , \tag{2.5.3}$$

provided that each member of (2.5.3) exists.

The generating function (2.5.3) corresponds to the main result of Kamarujjama et al. [46,p.361(1.8)] in which the k-summand should be corrected to read n in place of ∞. Yet another special case of the assertion (2.4.2) would occur when we set

$$\Omega\ (m, n, k) = \Omega'_m\ \Omega''_n\ \Omega'''_k\ , \tag{2.5.4}$$

so that the left-hand side of (2.4.2) becomes a product of three series with essentially arbitrary coefficients. On setting

$$\Omega'_m\ \frac{(-p)_{M_1 m}\ (a_i)_m}{(b_j)_m\ (1+v+M)_m}, \tag{2.5.5}$$

$$\Omega''_m\ \frac{(-q)_{M_2 m}\ (c_l)_n}{(d_s)_n\ (1+\mu+N)_n}, \tag{2.5.6}$$

and

$$\Omega'''_k\ \frac{(-r)_{M_3 k}\ (e_u)_k}{(f_v)_k\ (1+\eta+R)_k}, \tag{2.5.7}$$

and replacing y, z, and x by y^{-2}, z^{-2} and x^{-2} respectively, we get

$$\begin{aligned}
&{}_{M_1+I}F_{J+1}\left[\begin{matrix}\Delta(M_1;\ -p),\ (a_i); \\ 1+v-M,\ (b_j);\end{matrix}\ M_1^{M_1}y^{-2}\right]\ {}_{M_2+1}F_{s+1}\left[\begin{matrix}\Delta(M_2;\ -q),\ (c_l); \\ 1+\mu-N,\ (d_s);\end{matrix}\ M_2^{M_2}z^{-2}\right] \\
&{}_{M_3+U}F_{v+1}\left[\begin{matrix}\Delta(M_3;\ -r),(e_u); \\ 1+\eta-R,\ (f_v);\end{matrix}\ M_3^{M_3}(-xz/y)^{-2}\right] = \sum_{m=-\infty}^{\infty}\ \sum_{n=m^*}^{\infty} \frac{(-p)_{M_1 m}(-q)_{M_2 m}(a_i)_m (c_l)_n\ y^{-2m} z^{-2n}}{(1+v-m)_m (1+\mu-N)\ (b_i)_m (d_s)_n m! n!} \\
&{}_{M_1+M_3+I+S+U+2}F_{M_2+J+I+V+3}\left[\begin{matrix}\Delta(M_1;\ -p+M_1 m),\ \Delta(M_3;-r),\ (a_i)+m,\ 1-(d)-n,\ (e_u), \\ \Delta(M_2;\ 1+q-M_2 n),\ (b_j)+m,\ 1-(c_l)-n,\ (f_v),\ 1+n-R,\end{matrix}\right. \\
&\left.\begin{matrix}-\mu+N-n,\ -n; \\ 1+v-M+m,\ m+1;\end{matrix}\ M_1^{M_1}\ M_2^{M_2}\ M_3^{M_3}\ (-1)^{M_2+1-s}\ -x^{-2}\right]
\end{aligned} \tag{2.5.8}$$

Again, setting

$$\Omega'_m \frac{(-p)_{M_1m}\ (a_i)_m\ (\lambda_1+p)_{M_1m}}{(b_j)_m\ (1+v+M)_m}\ , \tag{2.5.9}$$

$$\Omega''_n \frac{(-q)_{M_2n}\ (c_l)_n\ (\lambda_2+q)_{M_2n}}{(d_s)_n\ (1+\mu+N)_n}\ , \tag{2.5.10}$$

and

$$\Omega'''_k \frac{(-\lambda)_{M_3k}\ (e_u)_k\ (\lambda_3+r)_{M_3k}}{(f_v)_k\ (1+\mu+R)_k}\ , \tag{2.5.11}$$

we obtain

$$ {}_{M_1+i}F_{J+1}\left[\begin{matrix}\Delta(M_1;-p),\ \Delta(M_1;\ \lambda_1+p),(a_i); \\ \qquad\qquad M_1^{2M_1}\ y \\ 1+v-M,\ (b_j);\end{matrix}\right]\ {}_{M_2+1}F_{s+1}\left[\begin{matrix}\Delta(M_2;-q),\Delta(M_2;\lambda_2+q),(c_l); \\ \qquad\qquad M_2^{2M_2}\ z \\ 1+\mu-N,\ (d_s);\end{matrix}\right]$$

$$ {}_{M_3+U}F_{v+1}\left[\begin{matrix}\Delta(M_3;\ -r),\ (m_3;\ \lambda_3+r),\ (e_u); \\ \qquad\qquad M_3^{2M_3}\ (-xz/y) \\ 1+\eta-R,\ (f_v);\end{matrix}\right]$$

$$=\sum_{m=-\infty}^{\infty}\ \sum_{n=m^*}^{\infty} \frac{(-p)_{M_1m}\ (-q)_{M_2m}\ (\lambda_1+p)_{M_1m}\ (\lambda_2+q)_{M_2m}\ (a_i)\ (c_l)_n\ y^m\ z^n}{(1+v-m)_m\ \ (1+\mu-N)\ \ (b_i)_m\ \ (d_s)_n\ \ m!\ n!}$$

$$ {}_{2M_1+2M_3+I+s+U+2}F_{2M_2+J+I+V+3}\left[\begin{matrix}\Delta(M_1;\ -p+M_1m),\Delta(M_1;\ \lambda_1+p+M_1m),\ \Delta(M_3;-r),\ \Delta(M_3;\ \lambda_3+r), \\ \\ \Delta(M_2;\ 1+q-M_2n),\ \Delta(M_2;1-\lambda_2-q-M_2n),\ (b_j)+m,\ 1-(c_l)-n,\end{matrix}\right.$$

$$\left.\begin{matrix}(a_i)+m,\ 1-(d_s)-n,\ (e_u),\ -\mu+N-n,\ -n; \\ \qquad\qquad M_1^{2M_1}\ M_2^{-2M_2}\ M_3^{2M_3}\ (-1)^{l-s-1}\ x \\ (f_v),\ 1+\eta-R,\ 1+v-M+m,\ m+1;\end{matrix}\right] \tag{2.5.12}$$

Note that (2.5.8) and (2.5.12) are given erroneously in [46]. It is obvious when both sides in [46, p.362(3.1) and p.363(3.2)] are not equal at x=0. If, in (2.5.4) we further let

$$\Omega'_m = \frac{(\alpha_1)_m \cdots\cdots (\alpha_p)_m}{(\beta_1)_m \cdots\cdots (\beta_q)_m}, \tag{2.5.13}$$

$$\Omega''_n = \frac{(\lambda_1)_n \cdots\cdots (\lambda_r)_n}{(\mu_1)_n \cdots\cdots (\mu_s)_n}, \tag{2.5.14}$$

and

$$\Omega'''_m = \frac{(\rho_1)_k \cdots\cdots (\rho_u)_k}{(\sigma_1)_k \cdots\cdots (\sigma_v)_k}, \tag{2.5.15}$$

we shall obtain the following hypergeometric generating function :

$${}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p;\\ \\ \beta_1,\ldots,\beta_q;\end{matrix}\; y\right] {}_rF_s\left[\begin{matrix}\lambda_1,\ldots,\lambda_r;\\ \\ \mu_1,\ldots,\mu_s;\end{matrix}\; z\right] {}_uF_v\left[\begin{matrix}\rho_1,\ldots,\rho_u;\\ \\ \sigma_1,\ldots,\sigma_v;\end{matrix}\; -\frac{xz}{y}\right]$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{\prod_{j=1}^{p} (\alpha_j)_m \prod_{j=1}^{r} (\lambda_j)_n}{\prod_{j=1}^{q} (\beta_j)_m \prod_{j=1}^{s} (\mu_j)_n} \frac{y^m}{m!} \frac{z^n}{n!}$$

$${}_{p+s+u+1}F_{q+r+v+1}\left[\begin{matrix}-n,\ (\alpha_p)+m,\ 1-(\mu_s)-n,\ (\rho_u);\\ \\ m+1,\ (\beta_q)+m,\ 1-(\lambda_r)-n,\ (\sigma_v);\end{matrix}\; (-1)^{r-s}\, x\right], \tag{2.5.16}$$

where, for convenience, $(\alpha_p)+m$ abbreviates the array of p parameters:

$$\alpha_1+m,\ldots\ldots,\alpha_p+m\ ,$$

with similar interpretations for $(\beta_q)+m$, etc.

The generating functions (2.5.8), (2.5.12) and (2.5.16) can also be deduced by appropriately specializing (2.5.3); indeed, except for some obvious notational variations, it is the main result of Pathan and Yasmeen [69, p. 3, Equation (1.5)].

Next we consider some multivariable applications of the assertion (2.4.4). First of all, by setting

$$\Omega\,(m,\ n,\ k) \equiv 1, \tag{2.5.17}$$

(2.4.4) immediately yields the following extension of Exton's generating function (2.1.4).

$$\exp\left(\sum_{j=1}^{r} (y_j + z_j - x_j z_j / y_j\right)$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \prod_{j=1}^{r} \left\{ {}_1F_1\,(-n_j;\ m_j+1;\ x_j)\,\frac{y_j^{m_j}}{m_j!}\ \frac{z_j^{n_j}}{n_j!} \right\} \tag{2.5.18}$$

If, in the assertion (2.4.4), we set

$$\Omega(m,n,k) = \Omega_1(m)\ \Omega_2(n)\ \Omega_3(k), \tag{2.5.19}$$

the left-hand side of (2.4.4) would reduce at once to a product of three multiple series with essentially arbitrary coefficients. Thus, by assigning suitable special values to the coefficients

$$\Omega_1(m),\ \Omega_2(n)\ \text{and}\ \ \Omega_3\ (k),$$

we can derive a number of generating functions involving the products of such multivariable hypergeometric functions as the familiar

Lauricella functions $F_A^{(r)}$, $F_B^{(r)}$, $F_C^{(r)}$, and $F_D^{(r)}$ of r variables [93, p.33] and their generalization introduced and studied by Srivastava and Daoust ([90] and [92]; see also [40]). In view of the multinomial expansion (cf. e.g. [93,p.329 (220)] :

$$(1-z_1-.....-z_r)^{-\lambda} = \sum_{m=0}^{\infty} (\lambda)_{m_1+......+m_r} \frac{z_1^{m_1}}{m_1!}......\frac{z_r^{m_r}}{m_r!} \tag{2.5.20}$$

$$(\lambda \in C; \quad |z_1+.......+z_r| < 1),$$

in (2.4.4) and (2.5.19)

$$\Omega_1\ (m) = \begin{cases} 1 & (m = 0) \\ 0 & (m \neq 0), \end{cases} \tag{2.5.21}$$

$$\Omega_2(n) = (\lambda_1)_{n_1}.......(\lambda_r)_{n_r}\ , \tag{2.5.22}$$

and

$$\Omega_3(k) = (\mu)_{k_1+.......+k_r}\ , \tag{2.5.23}$$

and replace x_j by $x_j y_j$ (j = 1,.....,r). We thus find for the Lauricella first function $F_A^{(r)}$ that (cf. [26,p.25(3.1)]; see also [94,p.494(8(i))].

$$(1-z_1)^{-\lambda_1}\ ......(1-z_r)^{-\lambda_r}\ (1+x_1z_1+......+x_rz_r)^{-\mu}$$

$$= \sum_{n=0}^{\infty} (\lambda_1)_{n_1}........(\lambda_r)_{n_r} \frac{z_1^{n_1}}{n_1!}......\frac{z_r^{n_r}}{n_r!}$$

$$.\ F_A^{(r)}\ [\mu,\ -n_1,....,\ -n_r;\ 1-\lambda_1-n_1,...,\ 1-\lambda_r-n_r;\ x_1,.....,\ -x_r]. \tag{2.5.24}$$

For the other three Lauricella functions, the assertion (2.4.4) [in conjunction with (2.5.19)] similarly yields the following generating functions:

$$(1-z_1-.....-z_r)^{-\lambda}\ (1+x_1z_1)^{-\mu_1}....(1+x_rz_r)^{-\mu_r}$$

$$= \sum_{n=0}^{\infty} (\lambda)_{n_1+.........+n_r} \quad \frac{z_1^{\,n_1}}{n_1!}.....\frac{z_r^{\,n_r}}{n_r!}$$

$$F_B^{(r)}\ [\mu_1,...,\mu_r,\ -n_1,...,-n_r;\ 1-n_1-...-n_r-\lambda;\ -x_1,....,-x_r\ ]\ , \tag{2.5.25}$$

$$(1-y_1-.....-y_r)^{-\lambda} \left(1+\frac{x_1}{y_1}+.........+\frac{x_r}{y_r}\right)^{-\mu}$$

$$= \sum_{m=-\infty}^{\infty} (\lambda)_{m_1+.........+m_r} \quad \frac{y_1^{\,m_1}}{m_1!}.....\frac{y_r^{\,m_r}}{m_r!}$$

$$.\ F_C^{(r)}\ [\mu,\ \lambda+m_1+....+m_r;\ m_1+1,.....,\ m_r+1;\ -x_1,....,-x_r\ ], \tag{2.5.26}$$

which provides a multivariable generalization of a known result [94,p.325(9)];

$$(\ 1-z_1-.....-z_r\ )^{-\lambda} \quad (\ 1+x_1z_1+....+x_rz_r\ )^{-\mu}$$

$$= \sum_{n=0}^{\infty} (\lambda)_{n_1+.........+n_r} \quad \frac{z_1^{\,n_1}}{n_1!}.....\frac{z_r^{\,n_r}}{n_r!}$$

$$.\ F_D^{(r)}\ [\mu,\ -n_1,...,-n_r;\ 1-n_1-...-n_r-\lambda;\ -x_1,....,-x_r\ ]. \tag{2.5.27}$$

If in (2.4.4) and (2.5.19), we set

$$\Omega_1(m) = (\lambda)_{m_1+....+m_r}\ , \tag{2.5.28}$$

$$\Omega_2(n) = (\mu)_{n_1+....n_r}\ , \tag{2.5.29}$$

and

$$\Omega_3(k) = (\nu)_{k_1+....+k_r}\ , \tag{2.5.30}$$

and then apply the multinomial expansion (2.5.20), we thus obtain the following multivariable generalization of another known result given recently by Pathan and Yasmeen [69,p.7 (3.5)] :

$$(1-y_1-\ldots\ldots-y_r)^{-\lambda}(1-z_1-\ldots\ldots-z_r)^{-\mu}\left(1+\frac{x_1z_1}{y_1}+\ldots\ldots+\frac{x_r\,z_r}{y_r}\right)^{-\nu}$$

$$=\sum_{m=-\infty}^{\infty}\ \sum_{n=m^*}^{\infty}(\lambda)_{m_1+\ldots+m_r}(\mu)_{n_1+\ldots\ldots+n_r}\prod_{j=0}^{r}\left\{\frac{y_j^{m_j}}{m_j!}\ \frac{z_j^{n_j}}{n_j!}\right\}$$

$$F^{2:1;\ldots;1}_{1:1;\ldots;1}\left[\begin{matrix}\nu,\ \lambda+m_1+\ldots+m_r:\ -n_1;\ldots\ldots;\quad -n_r;\\ 1\ -n_1-\ldots-n_r-\mu:\ m_1+1;\ldots;\ m_r+1;\end{matrix}\ -x_1,\ldots,-x_r\right],\qquad(2.5.31)$$

where $F^{2:1;\ldots;1}_{1:1;\ldots;1}$ is a special case of the multivariable hypergeometric fucntion defined by (1.11.1).

A confluent case of this last multivariable generating function is worthy of note. Indeed, upon replacing x_j on both sides of (2.5.31) by x_j/ν ($j=1,\ldots,r$), if we let $\nu\to\infty$, (8.5.31) readily yields

$$(1-y_1-\ldots\ldots-y_r)^{-\lambda}(1-z_1-\ldots\ldots-z_r)^{-\mu}\exp\left(-\frac{x_1z_1}{y_1}-\ldots\ldots-\frac{x_r\,z_r}{y_r}\right)$$

$$=\sum_{m=-\infty}^{\infty}\ \sum_{n=m^*}^{\infty}(\lambda)_{m_1+\ldots+m_r}(\mu)_{n_1+\ldots\ldots+n_r}\prod_{j=0}^{r}\left\{\frac{y_j^{m_j}}{m_j!}\ \frac{z_j^{n_j}}{n_j!}\right\}$$

$$F^{1:1;\ldots;1}_{1:1;\ldots;1}\left[\begin{matrix}\lambda+m_1+\ldots\ldots+m_r:\ -n_1;\ldots\ldots\ldots;\quad -n_r;\\ 1\ -n_1-\ldots-n_r-\mu:\ m_1+1;\ldots\ldots;\ m_r+1;\end{matrix}\ -x_1,\ldots,-x_r\right].\qquad(2.5.32)$$

which can also be deduced directly from (2.4.4) and (2.5.19) by replacing the choice (2.5.30) by

$$\Omega_3 (k) \equiv 1.$$

Formula (2.5.32) can be shown to reduce to the exponential generating function (2.5.18) if we first make the following variable changes:

$$x_j \to \lambda^{-1}\mu x_j\ ,\ y_j \to \lambda^{-1}y_j\ ,\quad \text{and}\quad z_j \to \mu^{-1}z_j\ (j = 1,\ldots,r),$$

and let min $(\lambda, \mu) \to \infty$.

Finally, we recall here a further generalization of the Hurwitz-Lerch Zeta function Φ (z,s,a) in the form (cf. [35,p.100(1.5)]) :

$$\Phi_\mu^* (z,s,a) = \sum_{n=0}^{\infty} \frac{(\mu)_n}{(n+1)^s} \frac{z^n}{n!} \tag{2.5.33}$$

$(a \neq 0, -1, -2,\ldots;\ \mu \in C;\ s \in C$ when $|z|<1$; $Re(s)>1$ when $|z| = 1)$,

so that, obviously,

$$\Phi_1^*(z,s,a) = \Phi(z,s,a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{2.5.34}$$

$(a \neq 0, -1, -2,\ldots;\ s \in C$ when $|z|<1$; $Re(s)>1$ when $|z|=1)$.

In fact, it readily follows from the definitions (2.5.31) and (2.5.34) that

$$\Phi^*_\mu(z,s,a)= \frac{1}{\Gamma(\mu)} \int_0^{\infty} t^{\mu-1}\, e^{-t}\, \Phi(zt,\ s,\ a)\, dt,\ (Re(\mu)>0), \tag{2.5.35}$$

provided that each member of (2.5.35) exist.

Equation (2.5.35) exhibits the fact that $\Phi^*_\mu(z,s,a)$ is essentially an Eulerian integral of the familiar function $\Phi(z,s,a)$. More interestingly, the main generating function for $\Phi^*_\mu(z,s,a)$, proven recently by Goyal and Laddha [35,p.101(2.4)] is a very specialized case of our result (2.4.4) when

$x_j = y_j - t_j = 0$ ($j=1,\ldots,r$), $z_1 = z$ and $z_j = 0$ ($j = 2,\ldots,r$), (2.5.36)

and

$$\Omega(m,n,p) = \Lambda\,(m_1,\ldots,m_r)\,\frac{(\mu)_n}{(a+n)^{\nu+m_1+\ldots+m_r}}\,, \tag{2.5.37}$$

where the multiple sequence $\{\Lambda(m_1,\ldots,m_r)\}$ is a suitably chosen quotient of Gamma functions [35,p.102(2.5)] and, for convenience,

$$n_1 = n.$$

The aforementioned generating function of Goyal and Laddha [35,p.101(2.4)] is merely a rewriting of an (r+1)–dimensional series as a single sum of the r-dimensional series involved. Moreover, in view of the elementary identity [94,p.52(3)] :

$$\sum_{m=0}^{\infty} f(m)\,\frac{z_1^{m_1}}{m_1!}\cdots\frac{z_r^{m_r}}{m_r!} = \sum_{m=0}^{\infty} f(m)\,\frac{(z_1+\ldots+z_r)^m}{m!}\,, \tag{2.5.38}$$

many of the multiple-series results, given by Goyal and Laddha [35], are no more general than the corresponding results involving a single series.

Much more general known families of multiple-series generating functions can be found reproduced (with proper credits) in the work of Srivastava and Manocha [94].

CHAPTER-III

ON CERTAIN CLASS OF BILINEAR AND BILATERAL GENERATING FUNCTIONS INVOLVING GENERALIZED POLYNOMIALS

3.1 INTRODUCTION

In 1972, Saran [76] gave two theorems on bilinear generating functions which are as follows :

Theorem A : Let

$$F(x, t) = \sum_{n=0}^{\infty} f_n(x)\, t^n,$$

where $f_n(x)$ is a polynomial of degree n in x, then

$$\frac{1}{\Gamma b} \int_0^{\infty} e^{-p}\, p^{b-1}\ {}_1F_1\,[c;\, b;\, yp/(y-1)]\ F(x, tp)\ dp$$

$$= (1-y)^c \sum_{n=0}^{\infty} (b)_n\ {}_2F_1\,[-n, c;\, b;\, y]\ f_n(x)\ t^n$$

provided the integral is convergent.

Theorem B : Let

$$G(x, t) = \sum_{n=0}^{\infty} g_n(x)\, t^n,$$

where $g_n(x)$ is a polynomial of degree n in x, then

$$\frac{1}{\Gamma a \, \Gamma b} \int_0^\infty \int_0^\infty e^{-(p+q)} \, p^{a-1} \, q^{b-1} \, {}_0F_1 [-; a; ypq/(1-y)] \; G(x, tpq/(1-y)) \, dp \, dq$$

$$= (1-y)^b \sum_{n=0}^{\infty} (a)_n \, (b)_n \;\; {}_2F_1 \; [-n, b+n; a; y] \;\; g_n(x) \, t^n$$

provided the integral is convergent.

In an attempt to obtain more general results on bilinear and bilateral generating functions, Caudhary [16] obtained three theorems involving a triple hypergeometric function $F^{(3)}$ [x, y, z] of Srivastava defined by (1.8.1) and Appell's hypergeometric functions F_2 and F_4 (cf. (1.4.2) and (1.4.4)). The Theorems obtained are of general character and include the above mentioned results of Saran and some other known results given earlier by Mathur [61], Brafman [8], Carlitz [12], Helim and Al-Salam [41], Manocha [58], Manocha and Sharma [59] and Srivastava [86] as particular cases.

This chapter aims at establishing three theorems involving multiple series with essentially arbitrary terms and which are generalizations of several known results of Chaudhary [16], Mathur [61], Saran [76], Srivastava [87], Manocha [58], Manocha and Sharma [59], Sharma and Mittal [79], Brafman [8], Gupta [38], Chaundy [17] and Saxena [77].

More interestingly, the main generating relation proven recently by Chaudhary [16, p.262(3.4)] is also corrected here.

3.2 MAIN THEOREMS

Theorem 1 : Let $C(k_1,\ldots,k_m)$ denotes a suitably bounded multiple sequences of arbitrary complex numbers for integer $m \geq 1$. Suppose also that $F(x,t)$ be a function having a formal power series expansion in t such that

$$F(x,t) = \sum_{n=0}^{\infty} C_n f_n(x) \quad t^n , \tag{3.2.1}$$

where $\{C_n\}_{n=0}^{\infty}$ is a sequence of parameters, independent of x and t, and $\{f_n(x)\}$ are polynomials of degree n in x. Then, for Re(q)>0, Re (p)>0, p and q being independent of x and t, such that the series $C(k_1,\ldots,k_m)$ and $F(x,t)$ remain uniformly convergent for $z \in (0,1)$,

$$\sum_{n=0}^{\infty} \frac{(p)_n}{(1+p-q)_n} C_n \sum_{k_1,\ldots,k_m=0}^{\infty} \frac{(p+n)_{k_1+\ldots+k_m}}{(q)_{k_1+\ldots+k_m}} C(k_1,\ldots,k_m) \prod_{j=1}^{m} \left(\frac{x_j^{k_j}}{k_j!}\right) f_n(x) \; t^n$$

$$= \frac{\Gamma q}{\Gamma p \, \Gamma q-p} \int_0^1 z^{p-1}(1-z)^{q-p-1} \sum_{k_1,\ldots,k_m=0}^{\infty} C(k_1,\ldots,k_m) \prod_{j=1}^{m} \left(\frac{x_j^{k_j} z}{k_j!}\right) \; F\left(x,\frac{tz}{z-1}\right)dz \tag{3.2.2}$$

provided that each side of (3.2.2) has a meaning.

Theorem 2 : Let $C(k_1,\ldots, k_m)$ denotes a suitably bounded multiple sequences of arbitrary complex numbers for integer $m \geq 1$. Suppose that $F(x,t)$ be a function constrained as in theorem 1. Then for Re(p)>0, p is independent of x and t, and s_j are positive integers, j=1, 2,...,m,

$$\sum_{n=0}^{\infty} (b)_n \; C_n \sum_{k_1,\ldots,k_{m=0}}^{\infty} (b+n)_{s_1k_1+\ldots+s_mk_m} \; C(k_1,\ldots,k_m) \prod_{j=1}^{m} \left(\frac{x_j^{k_j}}{k_j!}\right) f_n(x) \; t^n$$

$$= \frac{1}{\Gamma b} \int_0^{\infty} e^{-p} \, p^{b-1} \sum_{k_1,\ldots,k_m=0}^{\infty} C(k_1,\ldots,k_m) \prod_{j=1}^{m} \left(\frac{(x_j \; p^{s_j})^{k_j}}{k_j \; !}\right) F(x,tp) \; dp \qquad (3.2.3)$$

provided the integral is convergent and that both the sides of (2.2.3) exist.

<u>Theorem 3</u> : Let $C(k_1,\ldots,k_m)$ denotes a suitably bounded multiple sequences of arbitrary complex numbers for integer $m \geq 1$. Suppose also that $F(x,t)$ be a function constrained as in theorem 1. Then for Re $(p)>0$, $Re(q)>0$ and $Re(h)>0$, p, q and h being independent of x and t, and $r<m$,

$$\sum_{n=0}^{\infty} (b)_n (c)_n \sum_{k_1,\ldots,k_m=0}^{\infty} (a)_{k_1+\ldots+k_r} (b+n)_{k_{r+1}+\ldots+k_m} (c+n)_{k_1+\ldots+k_m} \; C(k_1,\ldots,k_m) \prod_{j=1}^{m} \left(\frac{x_j^{k_j}}{k_j!}\right) f_n(x) \; t^n$$

$$= \frac{1}{\Gamma a \Gamma b \, \Gamma c} \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-(P+q+h)} \, p^{a-1} q^{b-1} h^{c-1} \sum_{k_1,\ldots,k_m=0}^{\infty} C(k_1,\ldots,k_m) \prod_{j=1}^{r} \left(\frac{x_j ph}{k_j!}\right)^{k_j}$$

$$\prod_{j=1+r}^{m} \left(\frac{x_j qh}{k_j!}\right)^{k_j} \quad F(x,tqh) \; dp \; dq \; dh \; , \qquad (3.2.4)$$

provided that the integral is convergent.

<u>Proof of theorems 1 to 3</u> : To prove theorem 1, we replace $F(x,t)$ by its series (3.2.1) in the integrand of (3.2.2). Changing the order of integragtion and summation, which is permissible due to the uniform convergence of the series involved and evaluating the inner Euler integral, we arrive at the result (3.2.2). The proofs of theorems 2 and 3 are similar to that of theorem 1. The following particular cases of the above theorems are of much interest.

<u>Corollary 1</u> : For m=3 or else $x_4 = \ldots\ldots = x_m = 0$ and

$$C(k_1,k_2,k_3) = \frac{((a))_{k_1+k_2+k_3}\,((b))_{k_1+k_2}((b'))_{k_2+k_3}((b''))_{k_3+k_1}\;((c))_{k_1}((c'))_{k_2}\,((c''))_{k_3}}{((d))_{k_1+k_2+k_3}\,((e))_{k_1+k_2}((e'))_{k_2+k_3}((e''))_{k_3+k_1}\;((f))_{k_1}\,((f'))_{k_2}\,((f''))_{k_3}},$$

theorem 1 would reduce to a known result [16, p. 262 (3.2)] :

$$\sum_{n=0}^{\infty} \frac{(p)_n}{(1+p+q)_n}\, F^{(3)}\left[\begin{matrix} p+n, & (a)::(b);(b');(b''): & (c);(c');(c''); \\ & & & x_1,x_2,x_3 \\ q, & (d)::(e);(e');(e''); & (f);(f');(f''); \end{matrix}\right] f_n(x)\; t^n$$

$$= \frac{\Gamma q}{\Gamma p\; \Gamma q-p} \int_0^1 z^{p-1}\,(1-z)^{q-p-1}$$

$$F^{(3)}\left[\begin{matrix} (a)\;::\;(b);\;(b');\;(b''):\;(c);\;(c');\;(c''); \\ & x_1 z,\; x_2 z,\; x_3 z \\ (d)\;::\;(e);\;(e');\;(e''):\;(f);\;\;(f');\;(f''); \end{matrix}\right]$$

$$.\, F\left(x, \frac{tz}{z-1}\right)\; dz\,. \tag{3.2.5}$$

where $F^{(3)}$ [x, y, z] is the triple hypergeometric series of Srivastava defined by (1.8.1).

On other hand, for $C(k_1,k_2,k_3) = \dfrac{(e_1)_{k_1}(e_2)_{k_2}(e_3)_{k_3}}{(f_1)_{k_1}(f_2)_{k_2}(f_3)_{k_3}}$, theorem 1 reduces to a result due to Mathur (see e.g. [61, p. 222(2.2)] and [16,p.263(3.8)]):

$$\sum_{n=0}^{\infty} \frac{(p)_n}{(1+p-q)_n}\, F_A^{(3)}\; p+n,\; e_1,\; e_2,\; e_3;\; f_1;\; f_2;\; f_3;\; x_1,\; x_2,\; x_3]\; f_n(x)\quad t^n$$

$$= \frac{\Gamma q}{\Gamma p\;\Gamma q-p} \int_0^1 z^{p-1}\,(1-z)^{q-p-1}\;\; F_A^{(3)}\,[q,e_1,e_2,e_3;\; f_1,\; f_2,\; f_3;\; x_1 z,\; x_2 z,\; x_3 z]$$

$$.\, F\left(x, \frac{tz}{z-1}\right), \tag{3.2.6}$$

$$|\,x_1\,| + |\,x_2\,| + |\,x_3\,| < 1.,$$

where $F_A^{(3)}$ is Lauricella's function defined by (1.10.1). Similarly results for Lauricella's functions $F_B^{(3)}$, $F_C^{(3)}$ and $F_D^{(3)}$ (see (1.10.2) to (1.10.4) given by Mathur [61] appear as special cases of theorem 1.

<u>Corollary 2</u> : In theorem 2, if we set, m=2, $s_1=s_2=1$ and $C(k_1,k_2)=\dfrac{(a)_{k_1}(c)_{k_2}}{(d)_{k_1}(b)_{k_2}}$,

we get a known [16, p.262(3.4)] result in its corrected form

$$\sum_{n=0}^{\infty} (b)_n \; C_n \; F_2\,[\,b+n,\; a,c;d,b;z,y]\; f_n(x)\; t^n$$

$$= \frac{1}{\Gamma b} \int_0^{\infty} e^{-p}\, p^{b-1}\; {}_1F_1\,[a;d;\; zp]\; {}_1F_1\,[c;b;yp]\; F(x,tp)\; dp. \qquad (3.2.7)$$

<u>Corollary 3</u> : On setting r=0, or else $x_1=.....=x_r=0$, theorem 3 gives us an elegant result in the form

$$\sum_{n=0}^{\infty} (b)_n\, (c)_n\; C_n \sum_{k_1,\ldots,k_m=0}^{\infty} (b+n)_{k_1+\ldots+k_m}\; (c+n)_{k_1+\ldots+k_m} C(k_1,\ldots,k_m) \prod_{j=1}^{m} \left(\frac{x_j^{k_j}}{k_j!}\right) f_n(x)\; t^n$$

$$= \frac{1}{\Gamma b\; \Gamma c} \int_0^{\infty}\!\int_0^{\infty} e^{-(q+h)}\, q^{b-1} h^{c-1} \sum_{k_1,\ldots,k_m=0}^{\infty} C(k_1,\ldots,k_m) \prod_{j=1}^{m} \left(\frac{(x_j qh)^{k_j}}{k_j!}\right)$$

$$.\; F(x,tqh)\; dq\; dh\;, \qquad (3.2.8)$$

which for m=2, $C(k_1,k_2)= \dfrac{1}{(e)_{k_1}(f)_{k_2}}$, gives us another known result [16,p.262(3.6)]

$$\sum_{n=0}^{\infty} (b)_n\; (c)_n\; C_n \;\; F_4\,[b+n,\; c+n;\; e,\; f;\; x_1,\; x_2]\;\; f_n(x)\; t^n$$

$$= \frac{1}{\Gamma b\; \Gamma c} \int_0^{\infty}\!\int_0^{\infty} e^{-(q+h)}\; q^{b-1}\; h^{c-1}\;\; {}_0F_1\,[-;\; e;\; x_1 qh]$$

$${}_0F_1\,[-;\; f;\; x_2 qh]\; F(x,tqh)\; dq\; dh\;. \qquad (3.2.9)$$

<u>Corollary 4</u> : On taking m=1, $C(k) = 1/(b)_k$, $x=1/(x-1)$, $s_1=C_n=1$ in (3.2.3) and (3.2.8) and making use of Euler's first linear transformation [73, p.60 (4)].

$$ {}_2F_1\left[\begin{matrix} a,\ b; \\ \\ c; \end{matrix}\ z\right] = (1-z)^{-b}\ {}_2F_1\left[\begin{matrix} c-a,\ b; \\ \\ c; \end{matrix}\ \frac{-z}{1-z}\right], \tag{3.2.10} $$

$$ |\, z \,| < 1 \quad \text{and} \quad |\, z/(1-z) \,| < 1, $$

we get Saran theorems for bilinear generating functions [76, p.12-13], given in section (3.1).

3.3 APPLICATIONS OF THE THEOREMS

By assigning suitable values to the arbitrary coefficients $C(k_1,\ldots,k_m)$ and choosing special generating function of the form (3.2.1), theorems 1, 2 and 3 can be applied to deduced bilateral and bilinear generating functions involving a fairly variety of special functions of one and several variables; e.g., the classical Jacobi, Hermite, Rice, Bessel, the Lauricella hypergeometric functions $F_A^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$ of n-variables and their generalization introduced and studied by Srivastava and Daoust (cf. [91] and [92]). In this section, we will mention some interesting applications of theorems 1 to 3.

(i) Consider the generating relation [88, p.76 (3.1)] :

$$ \sum_{n=0}^{\infty} \frac{((c))_n((e))_n}{((d))_n\ ((f))_n}\ {}_{1+D+A}F_{C+B}\left[\begin{matrix} -n, 1-(d)-n, (a); \\ 1-(c)-n\quad ,(b); \end{matrix}\ x\right] \frac{t^n}{n!} $$

$$ = F^{E:A;C}_{F:B;D}\left[\begin{matrix} (e):(a);(c); \\ (f):(b);(d); \end{matrix}\ (-1)^{C-D+1}xt\ ,\ t\right]. \tag{3.3.1} $$

On taking $\mathcal{F}(x,t) = F^{E:A;C}_{F:B;D}\left[\begin{matrix}(e):(a);(c); \\ (f):(b);(d);\end{matrix}\ (-1)^{C-D+1}\ xt\ ,\ t\right]$,

where $F^{A:B;B'}_{B:D;D'}[x,y]$ is Kampe' de Fe'riet's double hypergeometric function [5,p.150] defined by (1.7.1), choosing

$$C(k_1,\ldots,k_m) = \frac{((u))_{k_1+\ldots+k_m}((g'))_{k_1}\ldots((g^{(m)}))_{k_m}}{((v))_{k_1+\ldots+k_m}((h'))_{k_1}\ldots((h^{(m)}))_{k_m}},$$

and combining (3.3.1) with (3.2.1), we get

$$\sum_{n=0}^{\infty}\frac{(p)_n((c))_n}{((d))}\ F^{U+1:G';\ldots;G^{(m)}}_{V+1:H';\ldots;H^{(m)}}\left[\begin{matrix}(p)+n,(u):(g');\ldots;(g^{(m)}); \\ q,(v)\ :\ (h');\ldots;(h^{(m)});\end{matrix}\ x_1,\ldots,x_m\right]{}_{1+D+A}F_{C+B}\left[\begin{matrix}-n,1-(d)-n,(a); \\ 1-(c)-n,(b);\end{matrix}\ x\right]t^n$$

$$=\sum_{s,k=0}^{\infty}\frac{(p)_{s+k}((a))_s((c))_k\ ((-1)^{C-D+1}xt)^s\ t^k}{((b))_s\ ((d))_k\ \ s!\ k!}$$

$$F^{U+1:G';\ldots;G^{(m)}}_{V+1:H;\ldots;H^{(m)}}\left[\begin{matrix}p+s+k,(u):(g');\ldots;(g^{(m)}); \\ q,(v)\ :(h');\ldots;(h^{(m)});\end{matrix}\ x_1,\ldots,x_m\right], \qquad (3.3.2)$$

where $F^{A:B';\ldots;B^{(n)}}_{C:D';\ldots;D^{(n)}}$ is the generalized Lauricella function defined by (1.11.1).

Equation (3.3.2) is reducible to generating functions involving Rice, Jacobi, Laguerre, Gegenbauer and Legendre polynomials. All these polynomials are closely associated with problems of applied nature.

For example the Laguerre polynomials $L_n^{(\alpha)}$ are deeply connected with problems involving the integration of Helmholtz's equation in parabolic coordinates, in the theory of the hydrogen atom etc. [57, p.76].

For instance, on taking A=B=C=D=1 and using the definition of Rice polynomials (1.14.14), equation (3.3.2) reduces to

$$\sum_{n=0}^{\infty} \frac{(p)_n}{(-\alpha-\beta)_n} F^{U+1:G';\ldots;G^{(m)}}_{V+1:H';\ldots;H^{(m)}} \left[\begin{matrix} p+n,(u):(g');\ldots;(g^{(m)}); \\ q,(v):(h');\ldots\ldots;(h^{(m)}); \end{matrix} x_1,\ldots,x_m \right] H_n^{(\alpha-n,\beta-n)}[\delta,\sigma, x]t^n$$

$$=\sum_{s,k=0}^{\infty} \frac{(p)_{s+k}(\delta)_s(-\alpha)_k \quad (xt)^s(-t^k)}{(\sigma)_s \ (-\alpha-\beta)_k \quad s! \ k!} F^{U+1:G';\ldots;G^{(m)}}_{V+1:H';\ldots;H^{(m)}} \left[\begin{matrix} p+s+k,(u):(g');\ldots;(g^{(m)}); \\ q,(v):(h');\ldots;(h^{(m)}); \end{matrix} x_1,\ldots,x_m \right], \qquad (3.3.3)$$

which for $x_3=\ldots=x_m=0$, gives us a known result [16,p.264(4.2)].

Again the generalized Kampe' de Fe'riet's function $F^{A:B';\ldots;B^{(m)}}_{C:D';\ldots;D^{(m)}}$ can be specialized to yield generating functions for Lauricella's $F_A^{(m)}$, Exton's ${}^{(k)}_{(1)}E_D^{(n)}$ and Chandel's ${}^{(k)}_{(1)}E_C^{(n)}$. For example, on setting $U=H'=\ldots=H^{(m)}=G'=\ldots=G^{(m)}=1$, $V=0$, and using the relationship between Rice and Jacobi polynomials (1.14.17), equation (3.3.3) reduces to

$$\sum_{n=0}^{\infty} \frac{(p)_n}{(-\alpha-\beta)_n} F_A^{(m)} [p+n, g_1,\ldots,g_m; h_1,\ldots,h_m; x_1,\ldots,x_m] P_n^{(\alpha-n,\beta-n)}(x) \quad t^n$$

$$= (\omega)^{-P} \quad F_A^{(m+1)} \left[p,g_1,\ldots; g_m,-\alpha;h_1,\ldots,h_m, -\alpha-\beta; \frac{x_1}{\omega},\ldots,\frac{x_m}{\omega}, \frac{-t}{\omega} \right], \qquad (3.3.4)$$

where $\omega = (1-((1-x)/2)t)$. If in (3.3.3), we set $U=2$, $V=G'=\ldots=G^{(m)}=0$, $H'=\ldots=H^{(m)}=1$ and use the relation (1.14.17), we obtain

$$\sum_{n=0}^{\infty} \frac{(p)_n}{(-\alpha-\beta)_n} F_C^{(m)} \left[p+n,u; h_1,\ldots,h_m; x_1,\ldots,x_m\right] P_n^{(\alpha-n,\beta-n)}(x) \quad t^n$$

$$= (\omega)^{-P} \ {}^{(k)}_{(1)}E_C^{(m+1)} \left[-\alpha, u,p; -\alpha-\beta,h_1,\ldots,h_m; \frac{x_1}{\omega},\ldots,\frac{x_m}{\omega}, \frac{-t}{\omega} \right] \qquad (3.3.5)$$

where ${}^{(k)}_{(1)}E_C^{(n)}$ is Chandel's function defined by (1.13.2).

In the special case U=V=H'=......=$H^{(m)}$ =0 and G'=...=$G^{(m)}$=1, of the relation (3.3.3), if we make use of (1.14.17), we shall obtain the generating relation

$$\sum_{n=0}^{\infty} \frac{(p)_n}{(-\alpha-\beta)_n} F_D^{(m)} \left[p+n, g_1,\ldots, g_m; q; x_1,\ldots,x_m\right] P_n^{(\alpha-n,\beta-n)}(x) \quad t^n$$

$$= \quad (\omega)^{-p} \, {}^{(1)}_{(1)}E_D^{(m+1)} \left[p, g_1,\ldots,g_n, -\alpha; q, -\alpha-\beta; \frac{x_1}{\omega},\ldots,\frac{x_m}{\omega}, \frac{-t}{\omega}\right], \qquad (3.3.6)$$

where ${}^{(k)}_{(1)}E_D^{(n)}$ is Exton's function defined by (1.13.1). On putting m=2, equations (3.3.4), (3.3.5) and (3.3.6) reduce to known results of Manocha [58, p.457(2.2.)], Sharma and Mittal [79,p.691(10)] and Manocha and Sharma [60,p.25] respectively.

It is to note that the Lauricella functions $F_A^{(n)}$ and $F_D^{(n)}$ have many applications in statistics, especially in statistical distributions [27, chapter 7].

(ii) Consider the generating relation [9, p.136(2)] :

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \mathcal{B}_n^m[(a);(b);x] \; t^n = (1-t)^{-\lambda} \; {}_{m+A}F_B\left[\begin{matrix}\Delta(m;\lambda) \; (a); \\ (b);\end{matrix} \; x \left(\frac{t}{t-1}\right)^m\right], \qquad (3.3.7)$$

where $\mathcal{B}_n^m$ [(a);(b);x] is Brafman's generalized hypergeometric polynomials defined by (1.14.18). On setting λ=1+p-q in (3.3.7), C(k)= $\dfrac{((u))_k((g))_k}{((v))_k\,((h))_k}$ and combining (3.3.7) with (3.2.2), we obtain

$$\sum_{n=0}^{\infty} \frac{(p)_n}{n!} \; {}_{1+U+G}F_{V+H+1}\left[\begin{matrix}p+n, \; (u), \; (g); \\ q,(v),(h);\end{matrix} \; z\right] \mathcal{B}_n^m \; [(a); \; (b); \; x] \quad t^n$$

$$= \sum_{s,k=0}^{\infty} \frac{((a))_s (p)_{ms}}{((b))_s \; s!} \left(x \; (-t/m)^m\right)^s F^{1:U+G\;;0}_{0:V+H+1;0}\left[\begin{matrix}p+ms:(u),(g),-;-; \\ \text{——}:(v),(h),q;-;\end{matrix} \; z, \; t\right]. \quad (3.3.8)$$

In the special case m=1 of (3.3.8), if we multiply both the sides by $e^u u^{-\beta}$ replace t by t/u and then evaluate the result obtained with the help of Hankel's contour integral [27,p.32(1.5.5.1)],

$$\int_c e^t\, t^{-c-n}\, dt = \frac{2\pi i}{\Gamma(c+n)}\,, \tag{3.3.9}$$

where n is a non-negative integer and c does not take non-positive integer values, we get,

$$\sum_{n=0}^{\infty} \frac{(p)_n}{(\beta)_n\, n!}\; {}_{1+U+G}F_{V+H+1}\left[\begin{matrix} p+n,(u),\ (g); \\ q,(v),(h); \end{matrix}\; z\right]\; {}_{1+A}F_B\left[\begin{matrix} -n,\ (a); \\ (b); \end{matrix}\; x\right] t^n$$

$$= F^{(3)}\left[\begin{matrix} p\ ::-;-;-:(u),(g)\ ;\ (a);-; \\ -::-;-;\beta:(v),(h),q;(b);-; \end{matrix}\quad z,\ -xt\ ,\ t\right]. \tag{3.3.10}$$

It is easy to observe that formula (3.3.10) is a generalization of known results [16,p.265(4.9) and p.266(4.10)] and [61,p.226(3.22)].

For m=1, A=G=H=U=V=0, B=1, equation (3.3.8) reduces to

$$\sum_{n=0}^{\infty} \frac{(p)_n}{(\alpha+1)}\; {}_1F_1\left[\begin{matrix} p+n; \\ q; \end{matrix}\; z\right] L_n^{(\alpha)}(x)\ t^n = (1-t)^{-p}\,\Psi_2\left[p;b,q;\ \frac{-xt}{1-t},\ \frac{z}{1-t}\right], \tag{3.3.11}$$

which for $z\to 0$, yield the well known result [94,p.132(5)]. If in (3.3.8) we set m=2, A=B=G=H=V=0, U=1 and replace x and t by $-1/x^2$ and 2xt respectively, it yield the divergent generating function

$$(1-z-2xt)^{-p}\; {}_2F_0\left[\tfrac{1}{2}p,\ \tfrac{1}{2}p+\tfrac{1}{2};-;\ \frac{4t^2}{(1-z-2xt)^2}\right] \cong (1-z)^{-p} \sum_{n=0}^{\infty} \frac{(p)_n}{n!} H_n(x) \left(\frac{t}{1-z}\right)^n,$$

(3.3.12)

where $H_n(x)$ is Hermite polynomials defined by (1.14.2). The Hermite polynomials play an important role in problems involving Laplace's equation in parabolic coordinates, in various problems in quantum mechanics and in probability theory (cf. [55] and [75]). For z=0, equation (3.3.12) reduces to result of Brafman [8,p.948(28)]. Formula (3.3.8) includes also other important special cases, for example, bilateral generating relation involving Gold-Hopper polynomials, Bessel polynomials [94,p.76(6) and p.75 (1)] and the Gegenbauer polynomials. The Gegenbauer polynomials generalizes the Legendre polynomials, which are important for spherical geometry [55, p.116].

(iii) Consider the generating function [8,p.947(27)]:

$$\sum_{n=0}^{\infty} {}_{1+A}F_C\left[\begin{matrix}-n, & (a); \\ & (c);\end{matrix}\ x\right]\frac{t^n}{n!} = e^t\ {}_AF_C\left[\begin{matrix}(a); \\ (c);\end{matrix}\ -xt\right] \tag{3.3.13}$$

If in (2.3), we take $F(x,t) = e^t\ {}_AF_C\left[\begin{matrix}(a); \\ (c);\end{matrix}\ -xt\right]$,

$$C(k_1,\dots k_m)= \frac{[(e):\alpha'_j;\dots;\alpha_j^{(m)}]\ [(g'):\phi'_j]\dots\dots\dots[(g^{(m)}):\ \phi_j^{(n)}]}{[(f):\beta'_j;\dots;\ \beta_j^{(m)}]\ [(h'):\Psi'_j]\dots\dots[(h^{(m)}):\Psi_j^{(m)}]},$$ and combine

(3.3.13) with (3.2.3) in theorem 2, we get

$$\sum_{n=0}^{\infty}\frac{(b)_n}{n!} F^{E+1:G';\dots;G^{(m)}}_{F:H';\dots\dots\dots;H^{(m)}}\left[\begin{matrix}[b+n:s_1,\dots,s_m],\ [(e):\alpha';\dots;\alpha^{(m)}]\ :\ [(g'):\phi'];\dots; \\ [(f):\beta';\dots;\beta^{(m)}]\ :\ [(h'):\Psi'];\dots;\end{matrix}\right.$$

$$\left.\begin{matrix}[(g^{(m)}):\phi^{(m)}]; \\ [(h^{(m)}):\Psi^{(m)}];\end{matrix}\ x_1,\dots,x_m\right]\quad {}_{1+A}F_C\left[\begin{matrix}-n, & (a) \\ & (c);\end{matrix}\ x\right]\quad t^n$$

$$= (1-t)^{-b}\, F^{E+1:G';\ldots;G^{(m)}}_{F\ :H';\ldots;H^{(m)}}\left[\begin{matrix}[b:s_1,\ldots,s_m,1],\ [(e):\alpha';\ldots;\alpha^{(m)}]:[(g'):\phi'];\ldots\ldots;[(g^{(m)}):\phi^{(m)}];\ [(a):1];\\ [(f):\beta';\ldots;\beta^{(m)}]:[(h'):\Psi'];\ldots;[(h^{(m)}):\Psi^{(m)}];\ [(c):1];\end{matrix}\right.$$

$$\left.\frac{x_1}{(1-t)^{s_1}},\ldots\ldots,\frac{x_m}{(1-t)^{s_m}},\quad \frac{-xt}{1-t}\right],\quad |\ t\ |<1, \tag{3.3.14}$$

For $x \to 0$, equation (3.3.14) reduces to a known result [87, p. 79(2.8)]

$$\sum_{n=0}^{\infty} \frac{(b)_n}{n!}\, F^{E+1:G';\ldots;G^{(m)}}_{F:H';\ldots\ldots;H^{(m)}}\left[\begin{matrix}[b+n:s_1,\ldots,s_m],\ [(e):\alpha';\ldots;\alpha^{(m)}]\ :\ [(g'):\phi'];\ldots;\\ [(f):\beta';\ldots;\beta^{(m)}]\ :\ [(h'):\Psi'];\ldots;\end{matrix}\right.$$

$$\left.\begin{matrix}[(g^{(m)}):\phi^{(m)}];\\ [(h^{(m)}):\Psi^{(m)}];\end{matrix}\ x_1,\ldots,x_m\right]\ t^n$$

$$= (1-t)^{-b}\, F^{E+1:G';\ldots;G^{(m)}}_{F\ :H';\ldots;H^{(m)}}\left[\begin{matrix}[b:s_1,\ldots,s_m],\ \ [(e):\alpha';\ldots;\alpha^{(m)}]:[(g'):\phi'];\ldots\ldots;[(g^{(m)}):\phi^{(m)}];\\ [(f):\beta';\ldots;\beta^{(m)}]:[(h'):\Psi'];\ldots;[(h^{(m)}):\Psi^{(m)}];\end{matrix}\right.$$

$$\left.\frac{x_1}{(1-t)^{s_1}},\ldots\ldots,\frac{x_m}{(1-t)^{s_m}}\right],\quad |\ t\ |<1, \tag{3.3.15}$$

which Srivastava [87] proves by using series rearrangement technique.

Further for $x_j=0$, $j=1,2,\ldots,m$, equation (3.3.14) yield a known result of Chaundy (cf.[17,p.62(25)] and [94, p.206(11)]).

$$\sum_{n=0}^{\infty} \frac{(b)_n}{n!}\ {}_{1+A}F_C\left[\begin{matrix}-n,\ (a);\\ (c);\end{matrix}\ x\right]\ t^n = (1-t)^{-b}\ {}_{1+A}F_C\left[\begin{matrix}b,(a);\\ (c);\end{matrix}\ \frac{-xt}{1-t}\right],\ |t|<1. \tag{3.3.16}$$

Next, on taking $x_2=\ldots=x_m=0$ and $s_1=\Psi'=\phi'=\alpha'=\beta'=1$, formula (3.3.14) gives us another known result [94,p.229(35)]

$$\sum_{n=0}^{\infty} \frac{(b)_n}{n!} \; {}_{1+A}F_C \left[\begin{matrix} -n, (a); \\ (c); \end{matrix} \; x \right] \; {}_{1+E}F_F \left[\begin{matrix} b+n, (e); \\ (f); \end{matrix} \; x_1 \right] \; t^n$$

$$= (1-t)^{-b} \; F^{1:A;E}_{0:C;F} \left[\begin{matrix} b: (a); (e); \\ -: (c); (f); \end{matrix} \; \frac{xt}{t-1}, \frac{x_1}{1-t} \right], \quad |t| < 1, \tag{3.3.17}$$

which originally obtained by using operational techniques.

(iv) Consider the Bedient generating function [94,p.186(48)]

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n} R_n (\beta,\gamma;x)\, t^n = F_2[\alpha,\beta,\beta;\ \gamma,\gamma;ut,vt] \;, \tag{3.3.18}$$

where $u = x - \sqrt{x^2-1}$, $v = x + \sqrt{x^2-1}$ and $R_n(\beta, \gamma; x)$ is Bedient's polynomial defined by (1.14.21).

Now, in theorem 2, choose $s_1 = \dots\dots = s_r = 2$, $s_{r+1} = \dots\dots = s_m = 1$,

$$C(k_1,\dots\dots,k_m) = \frac{(e_{r+1})_{k_{r+1}} \dots\dots (e_m)_{k_m}}{(f_1)_{k_1} \dots\dots (f_m)_{k_m}} \;,$$

and

$$f_n(x) = \frac{(\alpha)_n}{(\gamma)_n} R_n(\beta,\gamma;x) \text{ and } \boldsymbol{F}(x, t) = F_2[\alpha,\beta,\beta;\ \gamma,\gamma;ut,vt] \;,$$

then in the resulting expression replace t by t/u, multiply both the sides by $e^u\, u^{-\alpha}$ and use (3.3.9), to get

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n} \, {}^{(r)}H_4^{(m)} [b+n, e_{r+1},\dots, e_m;\ f_1,\dots, f_m;\ x_1,\dots,x_m] \quad R_n (\beta,\gamma;x)\, t^n$$

$$= {}^{(r)}H_4^{(m+2)} [b, e_{r+1},\dots,e_m, \beta, \beta;\ f_1,\dots, f_m, \gamma, \gamma;\ x_1,\dots,x_m, ut, vt]. \tag{3.3.19}$$

Indeed, equation (3.3.19) is reducible to a number of bilateral generating relations involving Horn's function x_8 [30,p.113] or $H_4^{(p)}$, Appell's F_2 and F_4, Lauricella's $F_A^{(n)}$ and $F_C^{(n)}$ and Gauss's ${}_2F_1$ defined by (1.12.6), (1.4.2), (1.4.4.), (1.10.1), (1.10.3) and (1.2.1) respectively. For instance on setting r = 0, equation (3.3.19) reduces to known result [16, p.266(4.12)]

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n} F_2\,[b+n,\ e_1,\ e_2;\ f_1,\ f_2;\ x_1,\ x_2]\quad R_n\,(\beta,\gamma;x)\ t^n$$

$$= F_A^{(4)}\,[b,\ e_1,\ e_2,\ \beta,\ \beta;\ f_1,\ f_2,\ \gamma,\ \gamma;\ x_1,\ x_2,\ ut,\ vt]. \qquad (3.3.20)$$

(v) Finally, we consider the generating function given by Bateman [73,p.256(1)] :

$$\sum_{n=0}^{\infty} \frac{t^n}{(\alpha+1)_n\ (\beta+1)_n} P_n^{(\alpha,\beta)}(x) = {}_0F_1\left[-;1+\alpha;\ \frac{t(x-1)}{2}\right] {}_0F_1\left[-;\beta+1;\frac{t(x+1)}{2}\right] \quad (3.3.21)$$

where $P_n^{(\alpha,\beta)}(x)$ denotes the Jacobi polynomials (cf. (1.14.8)).

In theorem 3, letting $C(k_1,\ldots,k_m) = \dfrac{1}{(e_1)_{k_1}\ldots(e_m)_{k_m}}$ and combining (3.2.4) with (3.3.21), we get a generating function involving Chandel's function ${}^{(K)}_{(1)}E_C^{(n)}$ defined by (1.13.2) :

$$\sum_{n=0}^{\infty} \frac{(b)_n\ (c)_n}{(\alpha+1)_n\ (\beta+1)_n}\ {}^{(r)}_{(1)}E_C^{(m)}[a,b+n,\ c+n;\ e_1,\ldots,e_m;\ x_1,\ldots x_m]\ P_n^{(\alpha,\beta)}(x)\ t^n$$

$$= {}^{(r)}_{(1)}E_C^{(m+2)}\ [a,b,c;\ e_1,\ldots,e_m,\alpha+1,\beta+1;\ x_1,\ldots x_m,\ \frac{t(x-1)}{2},\ \frac{t(x+1)}{2}\ ]. \qquad (3.3.22)$$

For r=0 or else $x_1=\ldots=x_r=0$, equation (3.3.22) reduces to a result due to Saxena (cf. [77,p.345] and [91, p.17]):

$$\sum_{n=0}^{\infty} \frac{(b)_n\ (c)_n}{(\alpha+1)_n\ (\beta+1)_n} F_C^{(m)}\left[b+n,\ c+n;\ e_1,\ldots,e_m;\ x_1,\ldots x_m\right]\ P_n^{(\alpha,\beta)}(x)\ t^n$$

$$= F_C^{(m+2)}\left[b,c;\ e_1,\ldots,e_m,\alpha+1,\beta+1;\ x_1,\ldots x_m,\ \frac{t(x-1)}{2},\ \frac{t(x+1)}{2}\ \right]. \qquad (3.3.23)$$

On putting r = 1 and m = 2, equation (3.3.22) yield the result

$$\sum_{n=0}^{\infty} \frac{(b)_n\ (c)_n}{(\alpha+1)_n(\beta+1)_n} F_2\ [c+n,a,b+n;\ e_1,\ e_2;\ x_1,\ x_2]\ P_n^{(\alpha,\beta)}(x)\ \ t^n$$

$$= \sum_{s=0}^{\infty} \frac{(a)_s\ (c)_s}{(e_1)_s\ \ s!} (x_1)^s$$

$$F_C^{(3)}\ \left[c+s,b;\ e_2,\ \alpha+1,\beta+1;\ x_2,\ \frac{t(x-1)}{2},\ \frac{t(x+1)}{2}\ \right]. \qquad (3.3.24)$$

When $x_2 \rightarrow 0$, in (3.3.24), we get a generating function involving Saran function F_E (cf. (1.13.9))

$$\sum_{n=0}^{\infty} \frac{(b)_n\ (c)_n}{(\alpha+1)_n(\beta+1)_n}\ {}_2F_1 \left[\begin{matrix} c+n,a; \\ e_1; \end{matrix}\ x_1\right]\ \ P_n^{(\alpha,\beta)}(x)\ \ t^n$$

$$= F_E\ \left[c,c,c,a,b,b;e_1,\ \alpha+1,\ \beta+1;\ x_1,\ \frac{t(x-1)}{2},\ \frac{t(x+1)}{2}\ \right], \qquad (3.3.25)$$

which is a known result due to Gupta [38, p.164(1)].

For $x_1 \rightarrow 0$, (3.3.22) reduces to a result of Manocha and Sharma

$$\sum_{n=0}^{\infty} \frac{(b)_n\ (c)_n}{(\alpha+1)_n\ (\beta+1)_n}\ {}_2F_1\ \left[b+n,\ c+n;\ e_2;\ x_2\right]\ \ P_n^{(\alpha,\beta)}(x)\ \ t^n$$

$$= F_C^{(3)}\ \left[b,c;\ e_2,\ \alpha+1,\beta+1;\ x_2,\ \frac{t(x-1)}{2},\ \frac{t(x+1)}{2}\ \right]\ , \qquad (3.3.26)$$

which was originally obtained by fractional derivative techniques.

On setting r = 3, m=4 and using the relation (1.13.11), equation (3.3.22) reduces to

$$\sum_{n=0}^{\infty} \frac{(b)_n\ (c)_n}{(\alpha+1)_n\ (\beta+1)_n} K_2\left[c+n,c+n,\ a,a,b+n;e_1,e_2,e_3,e_4;\ x_1,\ x_2,\ x_3,\ x_4\right] P_n^{(\alpha,\beta)}(x)$$

$$= {}^{(3)}_{(1)}E_C^{(6)}\ \left[a,b,c;\ e_1,e_2,e_3,e_4,\ \alpha+1,\beta+1;\ x_1,\ x_2\ x_3,\ x_4\ \tfrac{1}{2}t(x-1),\ \tfrac{1}{2}t(x+1)\right].$$

(3.3.27)

Similarly, on letting r=2, m=4 in (3.3.22) and using the relation (1.13.12), we get

$$\sum_{n=0}^{\infty} \frac{(b)_n\ (c)_n}{(\alpha+1)_n\ (\beta+1)_n}$$

$$K_5\ \left[c+n,\ c+n,\ c+n,\ c+n,\ a,a,b+n;e_1,e_2,e_3,e_4;\ x_1,\ x_2,\ x_3,\ x_4\right]\ P_n^{(\alpha,\beta)}(x)$$

$$= {}^{(2)}_{(1)}E_C^{(6)}\ \left[a,b,c;\ e_1,e_2,e_3,e_4,\ \alpha+1,\beta+1;\ x_1,\ x_2\ x_3,\ x_4,\ \tfrac{1}{2}t(x-1),\ \tfrac{1}{2}t(x+1)\ \right].$$

(3.3.28)

where K_2 and K_5 are Exton's quadruple hypergeometric functions defined by [27,p.90]

$$K_2\ [a,\ a,\ a,\ a;\ b,\ b,\ b,\ c;\ d_1,d_2,d_3,d_4;\ x,\ y,\ z,\ t]$$

$$= \sum_{m,n,k,p=0}^{\infty} \frac{(a)_{m+n+k+p}\ (b)_{m+n+k}\ (c)_p\ x^m\ y^n\ z^k\ t^p}{(d_1)_m\ (d_2)_n\ (d_3)_k\ (d_4)_p\ m!\ n!\ k!\ p!}$$

and

$$K_5\ [a,\ a,\ a,\ a;\ b_1,\ b_1,\ b_2,\ b_2;\ c_1,c_2,c_3,c_4;\ x,\ y,\ z,\ t]$$

$$= \sum_{m,n,k,p=0}^{\infty} \frac{(a)_{m+n+k+p}\ (b_1)_{m+n}\ (b_2)_{k+p}\ x^m\ y^n\ z^k\ t^p}{(c_1)_m\ (c_2)_n\ (c_3)_k\ (c_4)_p\ m!\ n!\ k!\ p!}$$

respectively.

CHAPTER-IV

ON DOUBLE GENERATING RELATIONS OF SINGLE HYPERGEOMETRIC POLYNOMIALS

4.1 INTRODUCTION

Exton [32] has introduced two interesting double generating relations of single hypergeometric polynomials involving the confluent hypergeometric function ${}_1F_1$ [54] and the generalized hypergeometric function ${}_AF_B$ defined by (1.2.1).

The purpose of this chapter is to introduce these generating functions as the main working tools to develop a theory of generating relations of special functions which are double generating relations of single polynomials. This chapter deals with a technique of integral operators for obtaining generating functions for Gauss hypergeometric function ${}_2F_1$, generalized hypergeometric functions ${}_AF_B$ and generalized Lauricella functions of n-variables $F^{A:B';\ldots;B^{(n)}}_{C:D';\ldots;D^{(n)}}$ defined by (1.2.1), (1.3.1) and (1.11.2) respectively.

Many known results of Exton [32], Chaundy [94,p.138(8)], Srivastava ([85] and [87]) and Srivastava and Manocha [94] are shown as special cases of these results. Besides treating these results, certain correct form of result of Exton [32, p.9 (5.10)] also follow as consequence of our results. Section 4.2 aims at obtaining generating relations for hypergeometric functions of single variable. Section 4.3 deals with special cases. In section 4.4, we further generalize results of section 4.2 to n-variables.

Finally, we mention in section 4.5 some special cases which involve Kampe' de Fe'riet function of two variables $F^{A:B;B'}_{C:D;D'}(x,y)$ [28, p.28 (1.4.3)], Appell's functions F_1, F_2 and F_3 [94,p.53] and Lauricella's functions $F_A^{(n)}$, $F_B^{(n)}$ and $F_D^{(n)}$ [94,p.60] .

4.2 DOUBLE GENERATING RELATIONS FOR HYPERGEOMETRIC FUNCTIONS OF SINGLE VARIABLE

Consider the generating functions for the confluent hypergeometric function ${}_1F_1$ and the generalized hypergeometric function ${}_AF_B$ [32, p.7(4.9) and p.11(6.5)]:

$$\sum_{m,n=0}^{\infty} \frac{(\alpha+1)_{m+n}\, x^m(-x)^n}{m!\, n!}\; {}_1F_1\left[\begin{matrix}-m; \\ \alpha+1;\end{matrix}\; z\right] {}_1F_1\left[\begin{matrix}-n; \\ \alpha+1;\end{matrix}\; y\right] = \exp(xy-xz)\; {}_0F_1\left[\begin{matrix}--; \\ \alpha+1;\end{matrix}\; -x^2yz\right], \quad (4.2.1)$$

$$\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}((u))_{m+n}((v))_{m+n}\, x^m(-x)^m}{((g))_{m+n}((e))_{m+n}((f))_{m+n}\, m!n!}\; {}_{E+V+Q+1}F_{U+F+P}\left[\begin{matrix}1-(e)-m-n,(v)+m+n,\ (q);\ -m \\ 1-(u)-m-n,\ (f)+m+n,\ (p);\end{matrix}\; (-1)^{1+E-U}\, y\right]$$

$$= {}_{D+2V+Q}F_{G+2F+P}\left[\begin{matrix}(d),\ (v/2),\ (v/2)+1/2,\ (q); \\ (g),\ (f/2),\ (f/2)+1/2,\ (p);\end{matrix}\; 4^{V-F}\, xy\right]. \quad (4.2.2)$$

In (4.2.1), if we replace z by zt, multiply both sides by $t^{l-1}e^{-pt}$ and take their Laplace transform with the help of the result [94,p.219(6)].

$$\int_0^{\infty} t^{\lambda-1} e^{-st}\, {}_AF_B\left[\begin{matrix}(a); \\ (b);\end{matrix}\; zt\right] = \lambda\, s^{-\lambda}\, {}_{A+1}F_B\left[\begin{matrix}(a),\ \lambda; \\ (b);\end{matrix}\; \frac{z}{s}\right], \quad (4.2.3)$$

$(Re(\lambda)>0,\ A\leq B;\ Re(s)>0$ if $A<B;\ Re(s)>Re(z)$ if $A=B)$, we obtain

$$\sum_{m,n=0}^{\infty} \frac{(\alpha+1)_{m+n}}{m!\, n!} x^m(-x)^n \; {}_2F_1\left[\begin{matrix} -m,\ l; \\ \alpha+1; \end{matrix}\; z\right] {}_1F_1\left[\begin{matrix} -n; \\ \alpha+1; \end{matrix}\; y\right]$$

$$= \exp(xy)\,(1+xz)^{-l}\, {}_1F_1\left[\begin{matrix} l; & \dfrac{-x^2zy}{(1+xz)} \\ \alpha+1; & \end{matrix}\right]. \qquad (4.2.4)$$

Next, if in (4.2.4), we replace y by yt, multiply both sides by $t^{r-1}\,e^{-Pt}$ and take Laplace transform with the help of (4.2.3), we get.

$$\sum_{m,n=0}^{\infty} \frac{(\alpha+1)_{m+n}}{m!\, n!} x^m(-x)^n \; {}_2F_1\left[\begin{matrix} -m,\ l; \\ \alpha+1; \end{matrix}\; z\right] {}_2F_1\left[\begin{matrix} -n,\ r; \\ \alpha+1; \end{matrix}\; y\right]$$

$$= (1-xy)^{-r}\,(1+xz)^{-l}\, {}_2F_1\left[\begin{matrix} l,\ r; & \dfrac{-x^2zy}{(1+xz)(1-xy)} \\ \alpha+1; & \end{matrix}\right]. \qquad (4.2.5)$$

Now starting from (4.2.5) and making use of the Laplace and inverse Laplace transform [25, p. 297(1)]

$$\mathcal{L}^{-1}\left\{ s^{-\lambda}\, {}_AF_B \begin{matrix} (a); & \dfrac{z}{s} \\ (b); & \end{matrix} : t \right\} = \frac{t^{\lambda-1}}{\lambda}\, {}_AF_{B+1}\left[\begin{matrix} (a); \\ (b),\ \lambda; \end{matrix}\; zt\right], \qquad (4.2.6)$$

$$(\operatorname{Re}(\lambda)>0, \qquad A \le B+1),$$

it is not difficult to show by induction that (cf. [84,p.305-311])

$$\sum_{m,n=0}^{\infty} \frac{(\alpha+1)_{m+n}}{m!\, n!} x^m(-x)^n \; {}_{1+L}F_{k+1}\left[\begin{matrix} -m,\ (l); \\ \alpha+1,(k); \end{matrix}\; z\right] {}_{1+R}F_{w+1}\left[\begin{matrix} -n,\ (r); \\ \alpha+1,(w); \end{matrix}\; y\right]$$

$$= \sum_{s=0}^{\infty} \frac{((l))_s((r))_s(-x^2yz)^s}{((k))_s((w))_s(\alpha+1)_s\, s!}\; {}_LF_K\left[\begin{matrix} (l)+s; \\ (k)+s; \end{matrix}\; -xz\right] {}_RF_W\left[\begin{matrix} (r)+s; \\ (w)+s; \end{matrix}\; xy\right]. \qquad (4.2.7)$$

Similarly, in case of equation (4.2.2), if we use the same method of proof of formula (4.2.7), we get

$$\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}((u))_{m+n}((v))_{m+n}((h))_{m+n}\ x^m(-x)^n}{((g))_{m+n}((e))_{m+n}((f))_{m+n}((t))_{m+n}\ m!\ n!}$$

$${}_{E+V+Q+A+1}F_{U+F+P+B}\left[\begin{matrix}1-(e)-m-n,\ (v)+m+n,\ (q),\ (a),\ -m; \\ 1-(u)-m-n,\ (f)+m+n,\ (p),\ (b);\end{matrix}\ (-1)^{1+E-V}\,y\right]$$

$$= {}_{D+2V+Q+A+H}F_{G+2F+P+B+T}\left[\begin{matrix}(d),(v/2),\ (v/2)+1/2,\ (q),\ (a),\ (h); \\ (g),\ (f/2),\ (f/2)+1/2,\ (p),\ (b),\ (t);\end{matrix}\ 4^{V-F}\ xy\right]. \qquad (4.2.8)$$

4.3 SPECIAL CASES

Now we mention some interesting special cases of the equations (4.2.5), (4.2.7) and (4.2.8). On setting $r=\alpha+1$, in (4.2.5), we get

$$\sum_{m=0}^{\infty} \frac{(\alpha+1)_m}{m!}\left[\frac{x}{1+x(1-y)}\right]^m {}_2F_1\left[\begin{matrix}-m,\ \ell; \\ \alpha+1;\end{matrix}\ z\right] = (1-xy)^{\ell-(\alpha+1)}(1-xy+xz)^{-\ell}(1+x-xy)^{\alpha+1} \qquad (4.3.1)$$

which for $y=1$ reduces to a known result [94,p.293(12)]. For $y=0$, (4.2.7) reduces to

$$(1+x)^{-\alpha-1}\sum_{m=0}^{\infty} \frac{(\alpha+1)_m}{m!}\left[\frac{x}{1+x}\right]^m {}_{1+L}F_{k+1}\left[\begin{matrix}-m,(\ell); \\ \alpha+1,(k);\end{matrix}\ z\right] = {}_LF_k\left[\begin{matrix}(\ell); \\ (k);\end{matrix}\ -xz\right]. \qquad (4.3.2)$$

If in (4.3.2), we put $L=L'+1$, $\ell_{L'+1}=\alpha+1$, we then have a known result [24,p.267]

$$\sum_{m=0}^{\infty} \frac{(\alpha+1)_m}{m!}\left[\frac{x}{1+x}\right]^m {}_{1+L}F_k\left[\begin{matrix}-m,(\ell); \\ (k);\end{matrix}\ z\right] = (1+x)^{\alpha+1}\ {}_{L+1}F_k\left[\begin{matrix}(\ell),\ \alpha+1; \\ (k);\end{matrix}\ -xz\right].$$

Further, if in (4.2.7), we put K=W=0, L=R=1, $l_1=r_1=1/2$ and $\alpha=0$, then it reduces to another known result due to Exton [32,p.9(5.10)] in its corrected form

$$\sum_{m,n=0}^{\infty} \frac{(m+n)!}{m!\,n!} \left[\frac{x}{z+(z^2-1)^{1/2}}\right]^m \left[\frac{-x}{y+(y^2-1)^{1/2}}\right]^n P_m(z)\ P_n(y)$$

$$= (1+x\zeta_1)^{-1/2}(1-x\zeta_2)^{-1/2} \quad 2/\pi\ K(\sqrt{\xi}), \tag{4.3.3}$$

where $\zeta_1 = \dfrac{2(z^2-1)^{1/2}}{z+(z^2-1)^{1/2}}$, $\zeta_2 = \dfrac{2(y^2-1)^{1/2}}{y+(y^2-1)^{1/2}}$,

$P_m(x)$ is Legendre polynomials defined by [32,p.8(5.6)]

$$P_m(x)=[x+(x^2-1)^{1/2}]^m\ {}_2F_1[-m,\ 1/2;\ 1;\ \zeta],\ \zeta = \frac{2(x^2-1)^{1/2}}{x+(x^2-1)^{1/2}} ,$$

$K(\xi)$ is the complete elliptic integral of first kind [23,p.318(5)] and $\xi = \dfrac{-x^2\zeta_1\zeta_2}{(1+x\zeta_1)(1-x\zeta_2)}$

If we replace x by $\dfrac{x}{(1-x)}$ in (4.2.8) together with V=U=E=F=D=G=Q=P=T=0 and H=1, we get

$$\sum_{m=0}^{\infty} \frac{(h)_m}{m!}\ {}_{1+A}F_B\left[\begin{matrix}-m,(a);\\ (b);\end{matrix}\ y\right] x^m = (1-x)^{-h}\ {}_{1+A}F_B\left[\begin{matrix}h,(a);\\ (b);\end{matrix}\ \frac{-xy}{(1-x)}\right], \tag{4.3.4}$$

$|x|<1$, which is a known result due to Chaundy [94,p.138(8)].

For E=U=V=F=0, the left hand side of (4.2.8) becomes separable in the form

$$\sum_{m=0}^{\infty} \frac{((d))_m((h))_m}{((g))_m((t))_m}\ {}_{D+H}F_{G+T}\left[\begin{matrix}(d)+m,(h)+m;\\ (g)+m,(t)+m;\end{matrix}\ -x\right] {}_{Q+A+1}F_{P+B}\left[\begin{matrix}(q),(a),-m;\\ (p),(b);\end{matrix}\ -y\right]\frac{x^m}{m!}$$

$$= {}_{D+Q+A+H}F_{G+P+B+T}\left[\begin{matrix}(d),(q),(a),(h);\\ (g),(p),(b),(t);\end{matrix}\ xy\right]. \tag{4.3.5}$$

Now, on putting A=B=H=T=0, (4.3.5) reduces to a known result due to Exton [32,p.11(6.6)]. Next, we turn to some generating functions involving the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ [73,p.254(1)] defined by (1.14.7). On setting B=E=U=V=Q=P=0, F=A=1, $a_1=\lambda$ and replacing y by -y, equation (4.2.8) reduces to

$$\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}((h))_{m+n}(f+m+n)_m x^m(-x)^n}{((g))_{m+n}((t))_{m+n}(f)_{2m+n}\, m!\, n!} \; {}_2F_1\left[\begin{matrix} -m,\ \lambda; \\ f+m+n; \end{matrix}\; y\right]$$

$$= {}_{1+D+H}F_{G+T+2}\left[\begin{matrix} (d),\ (h),\ \lambda; \\ (g),(t),f/2,f/2+1/2; \end{matrix}\; \frac{-xy}{4}\right]. \qquad (4.3.6)$$

For f= $-\alpha-\beta$, $\lambda= -\alpha$, equation (4.3.6) yields an interesting generating function for Jacobi polynomials given by

$$\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}((h))_{m+n}(-x)^n}{((g))_{m+n}((t))_{m+n}(-\alpha-\beta)_{2m+n}} \left[\frac{2x}{1-y}\right]^m P_m^{(\alpha-m,\ \beta-2m-n)}(y)$$

$$= {}_{1+D+H}F_{G+T+2}\left[\begin{matrix} (d),\ (h),\ -\alpha; \\ (g),(t),\ (-\alpha-\beta)/_2,\ (-\alpha-\beta)/_2+(1/2)\ ; \end{matrix}\; \frac{-2x}{4(1-y)}\right], \qquad (4.3.7)$$

which follows from [85,p.593(15)]

$$P_n^{(\alpha-n,\ \beta-n)}(x) = \binom{n-\alpha-\beta-1}{n}\left(\frac{1-x}{2}\right)^n {}_2F_1\left[\begin{matrix} -n,\ -\alpha; \\ -\alpha-\beta; \end{matrix}\; \frac{2}{1-x}\right]. \qquad (4.3.8)$$

On replacing z and y respectively by $\left(\frac{1-z}{2}\right)$ and $\left(\frac{1-y}{2}\right)$ in (4.2.5), setting l = r=1+α+β and using the definition [85, p.593(20)]..

$$P_n^{(\alpha,\ \beta)}(x) = \binom{\alpha+n}{n} {}_2F_1\left[\begin{matrix} -n,\ 1+\alpha+\beta+n; \\ 1+\alpha\ ; \end{matrix}\; \frac{1-x}{2}\right], \qquad (4.3.9)$$

we get

$$\sum_{m,n=0}^{\infty} \frac{(\alpha+1)_{m+n}\, x^m(-x)^n}{(\alpha+1)_m\,(\alpha+1)_n}\, P_m^{(\alpha,\beta-m)}(z)\, P_n^{(\alpha,\beta-n)}(y) = \{(1-x/2+xy/2)(1+x/2-xz/2)\}^{-(1+\alpha+\beta)}$$

$$ {}_2F_1\left[\begin{matrix}1+\alpha+\beta,\ 1+\alpha+\beta; \\ \alpha+1;\end{matrix}\ \frac{-x^2(1-z)(1-y)}{(2+x-xz)(2-x+xy)}\right] \tag{4.3.10}$$

4.4 DOUBLE GENERATING RELATIONS FOR HYPERGEOMETRIC FUNCTIONS OF SEVERAL VARIABLES

We shall now generalize some relations of section 4.2 and we will show how Laplace and inverse Laplace transforms of equations (4.2.7) and (4.2.2) would yield a generating functions of several variables, which are double generating relations.

We recall the definition of the generalised Kampe' de Fe'rier function of several variables [28,p.28(1.4.3)] given in (1.11.2).

$$F^{A:B';\ldots;B^{(n)}}_{C:D';\ldots;D^{(n)}}[x_1,\ldots,x_n] = F^{A:B';\ldots;B^{(n)}}_{C:D';\ldots;D^{(n)}}\left[\begin{matrix}(a):(b');\ldots;(b^{(n)}); \\ (c):(d');\ldots;(d^{(n)});\end{matrix}\ x_1,\ldots,x_n\right]$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{((a))_{m_1+\ldots+m_n}((b'))_{m_1}\ldots((b^{(n)}))_{m_n}\ x_1^{m_1}\ldots\ldots x_n^{m_n}}{((c))_{m_1+\ldots+m_n}\ ((d'))_{m_1}\ldots((d^{(n)}))_{m_n}\ m_1!\ldots\ldots m_n!} \tag{4.4.1}$$

On multiplying both sides of equation (4.2.7) by

$$t^{a_1-1}\ e^{-st}\ F^{A:B';\ldots;B^{(n)}}_{C:D';\ldots;D^{(n)}}[z_1t,\ldots,z_nt],$$

replacing z by zt and taking Laplace transform with the help of (4.2.3), we obtain

$$\sum_{m,n=0}^{\infty} \frac{(\alpha+1)_{m+n}x^m(-x)^n}{m!\,n!} \sum_{m_1,\ldots,m_n=0}^{\infty} \Omega(m_1,\ldots,m_n)\ {}_{2+L}F_{k+1}\left[\begin{matrix}a_1+m_1+\ldots+m_n,\ (\ell),-m; \\ \alpha+1,\ (k);\end{matrix}\ z\right]$$

$${}_{1+R}F_{W+1}\left[\begin{matrix}-n,(r);\\ \alpha+1,(w);\end{matrix}\ y\right] = \sum_{m_1,\dots,m_n=0}^{\infty} \Omega(m_1,\dots,m_n) \sum_{s=0}^{\infty} \frac{(a_1+m_1+\dots+m_n)_s((\ell))_s((r))_s(-x^2yz)^s}{((k))_s((w))_s\,(\alpha+1)_s \quad s!}$$

$${}_{1+L}F_K\left[\begin{matrix}a_1+m_1+\dots+m_n+s,(\ell)+s;\\ (k)+s;\end{matrix}\ -xz\right] {}_RF_W\left[\begin{matrix}(r)+s;\\ (w)+s\,;\end{matrix}\ xy\right], \tag{4.4.2}$$

where $\Omega(m_1,\dots,m_n) = \dfrac{((a))_{m_1+\dots+m_n}((b'))_{m_1}\dots((b^{(n)}))_{m_n}\, z_1^{m_1}\dots z_n^{m_n}}{((c))_{m_1+\dots+m_n}((d'))_{m_1}\dots((d^{(n)}))_{m_n} \quad m_1!\dots m_n!}$.

Now, replacing y by yt in (4.3.2) and multiplying both the sides by

$$t^{u_1-1}e^{-st}F^{U:E';\dots;E^{(r)}}_{V:F';\dots;F^{(r)}}[y_1t,\dots,y_rt],$$

and taking the Laplace transform with the help of (4.2.3), we get

$$\sum_{m,n=0}^{\infty}\frac{(\alpha+1)_{m+n}x^m(-x)^n}{m!\,n!}\sum_{m_1,\dots,m_n,K_1,\dots,K_r=0}^{\infty}\Omega(m_1,\dots,m_n)\wedge(k_1,\dots,k_r)\,{}_{2+L}F_{k+1}\left[\begin{matrix}a_1+m_1+\dots+m_n,\ -m,(\ell);\\ \alpha+1\ \ ,(k);\end{matrix}\ z\right]$$

$${}_{2+R}F_{W+1}\left[\begin{matrix}u_1+k_1+\dots+k_r,\ -n,\ (r);\\ \alpha+1,\qquad (w);\end{matrix}\ y\right] = \sum_{m_1,\dots,m_n,k_1,\dots,k_r=0}^{\infty}\Omega(m_1,\dots,m_n)\wedge(k_1,\dots,k_r)\sum_{s=0}^{\infty}\frac{(a_1+m_1+\dots+m_n)_s}{((k))_s((w))_s}$$

$$\frac{(u_1+k_1+,\dots,k_r)_s\,((\ell))_s((r))_s\,(-x^2yz)^s}{(\alpha+1)_s \qquad s!}\,{}_{1+L}F_k\left[\begin{matrix}a_1+m_1+\dots+m_n+s,(\ell)+s;\\ (k)+s;\end{matrix}\ -xz\right]{}_{1+R}F_w\left[\begin{matrix}u_1+k_1+\dots+k_r+s,(r)+s;\\ (w)+s;\end{matrix}\ xy\right], \tag{4.4.3}$$

where $\wedge(k_1,\dots,k_r) = \dfrac{((u))_{k_1+\dots+k_r}((e'))_{k_1}\dots((e^{(r)}))_{k_r}\,y_1^{k_1}\dots y_r^{k_r}}{((v))_{k_1+\dots+k_r}((f'))_{k_1}\dots((f^{(r)}))_{k_r} \quad k_1!\dots k_r!}$.

Now starting from (4.4.3) and making use of inverse Laplace and Laplace transform with the help of the results (4.2.3) and (4.2.6), the method of mathematical induction and the relation $(a)_{m+n}=(a)_m(a+m)_n$, we obtain on obvious simplification

$$\sum_{m,n=0}^{\infty} \frac{(\alpha+1)_{m+n} x^m (-x)^n}{m!\ n!} \; F^{A:B';\ldots;B^{(n)};\,L+1}_{C:D';\ldots;D^{(n)};\,K+1}\left[\begin{matrix}(a) : (b');\ldots;\ (b^{(n)});\ (\ell),\ -m; \\ (c) : (d');\ldots;\ (d^{(n)});\ (k),\ \alpha+1;\end{matrix}\ z_1,\ldots,z_n,\ z\right]$$

$$F^{U:E';\ldots;E^{(r)};\,R+1}_{V:F';\ldots;F^{(r)};\,W+1}\left[\begin{matrix}(u):(e');\ldots;(e^{(r)});(r),-n; \\ (v) : (f');\ldots;\ (f^{(r)});\ (w),\ \alpha+1;\end{matrix}\ y_1,\ldots,y_r,\ y\right]$$

$$= \sum_{s=0}^{\infty} \frac{((\ell))_s\,((r))_s\,((a))_s\,((u))_s\,(-x^2yz)^s}{((k))_s\,((w))_s((c))_s\,((v))_s\,(\alpha+1)_s\ s!}$$

$$F^{A:B';\ldots;B^{(n)};\,L}_{C:D';\ldots;D^{(n)};\,K}\left[\begin{matrix}(a)+s : (b');\ldots;\ (b^{(n)});\ (\ell)+s; \\ (c)+s : (d');\ldots;\ (d^{(n)});\ (k)+s;\end{matrix}\ z_1,\ldots,z_n,\ -xz\right]$$

$$F^{U:E';\ldots;E^{(r)};\,R}_{V:F';\ldots;F^{(r)};\,W}\left[\begin{matrix}(u)+s : (e');\ldots\ldots;(e^{(r)}),\ (r)+s; \\ (v)+s : (f');\ldots;\ (f^{(r)});\ (w)+s;\end{matrix}\ y_1,\ldots,y_r,\ xy\right]. \tag{4.4.4}$$

Similarly on multypling both the sides of (4.2.2) by

$$t^{a_1-1}\,e^{-st}\,F^{A:B';\ldots;B^{(n)}}_{C:D';\ldots;D^{(n)}}\ [x_1t,\ldots,\ x_nt]\ ,$$

replacing y by yt, using Laplace transforms formula (4.2.3) and evaluating the integrals on both the sides and then multiplying both the sides of the resulting expression by $t^{h-1}\,e^{-st}$, replacing x by xt and again taking the Laplace transform with the help of (4.2.3), we obtain

$$\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}((u))_{m+n}((v))_{m+n}(h)_{m+n} x^m(-x)^n}{((g))_{m+n}((e))_{m+n}((f))_{m+n}\ m!\ n!} \sum_{m_1,\ldots,m_n=0}^{\infty} \Phi(m_1,\ldots,m_n)$$

$${}_{E+V+Q+2}F_{U+F+P}\left[\begin{matrix}1-(e)-m-n,\ (v)+m+n,\ (q),\ a_1+m_1+\ldots+m_n,\ -m; \\ 1-(u)-m-n,\ (f)+m+n,\ (p);\end{matrix}\ (-1)^{1+E-U}y\right] = \sum_{m_1,\ldots,m_n=0}^{\infty} \Phi(m_1,\ldots,m_n)$$

$${}_{D+2V+Q+2}F_{G+2F+P}\left[\begin{matrix}(d),(v/2),(v/2)+1/2,\ (q),a_1+m_1+\ldots+m_n,\ h; \\ (g),\ (f/2),(f/2)+1/2,(p);\end{matrix}\ 4^{V-F}xy\right], \tag{4.4.5}$$

$$\text{where } \Phi(m_1,\ldots,m_n) = \frac{((a))_{m_1+\ldots+m_n}((b'))_{m_1}\ldots((b^{(n)}))_{m_n}}{((c))_{m_1+\ldots+m_n}((d'))_{m_1}\ldots((d^{(n)}))_{m_n}} \frac{x_1^{m_1}\ldots\ldots x_n^{m_n}}{m_1!\ldots\ldots m_n!}.$$

Again, starting from (4.3.5) and making use of Laplace and inverse Laplace transform techniques with the help of the results (4.2.3) and (4.2.6) and the method of mathematical induction, we obtain.

$$\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}((u))_{m+n}((v))_{m+n}((h))_{m+n}\, x^m(-x)^n}{((g))_{m+n}((e))_{m+n}((f))_{m+n}\,((t))_{m+n}\;\; m!\; n!}$$

$$F^{A:B';\ldots;B^{(n)};\,E+V+Q+1}_{C:D';\ldots;D^{(n)};\,U+F+P}\left[\begin{matrix}(a):(b');\ldots;(b^{(n)});\ 1-(e)-m-n,\ (v)+m+n,\ (q),\ -m;\\ (c):(d');\ldots;(d^{(n)});\ 1-(u)-m-n,(f)+m+n,(p);\end{matrix}\ x_1,\ldots,x_n,(-1)^{1+E-U}y\right]$$

$$=F^{A:B';\ldots;B^{(n)};\,D+2V+Q+H}_{C:D';\ldots;D^{(n)};\,G+2F+P+T}\left[\begin{matrix}(a):(b');\ldots;(b^{(n)});\ (d),(v/2),(v/2)+1/2,\ (q),(h);\\ (c):(d');\ldots;\ (d^{(n)});\ (g),(f/2),(f/2)+1/2,\ (p),(t);\end{matrix}\ x_1,\ldots,x_n,\ 4^{V-E}xy\right]. \tag{4.4.6}$$

4.5 SPECIAL CASES

It is easy to observe that the equations (4.4.4) and (4.4.6) give a large number of generating functions, new as well as known. In this section, we will mention only some special cases of our formulas (4.4.4) and (4.4.6). The well known Kampe' de Fe'riet function of two variables is defined and represented in the following manner (cf. (1.7.1))

$$F^{A:B;B'}_{C:D;D'}[x,y] = F^{(2)}\left[\begin{matrix}(a):(b);(b');\\ (b):(d);(d');\end{matrix}\ x,y\right] = \sum_{m,n=0}^{\infty} \frac{((a))_{m+n}((b))_m((b'))_n\, x^m y^n}{((c))_{m+n}((d))_m((d'))_n\;\; m!\, n!} \tag{4.5.1}$$

By specializing the various parameters in (4.4.4) and (4.4.6) to suit case (4.5.1) above , we obtain the generating functions

$$\sum_{m,n=0}^{\infty} \frac{(\alpha+1)_{m+n}}{m!\ n!} x^m(-x)^n \quad F^{(2)}\left[\begin{matrix}(a) : (\ell), -m; (b); \\ (c) : (k), 1+\alpha;(d);\end{matrix} z,t\right] F^{(2)}\left[\begin{matrix}(u) : (r), -n; (e); \\ (v):(w), 1+\alpha; (f);\end{matrix} y,\omega\right]$$

$$= \sum_{s=0}^{\infty} \frac{((\ell))_s ((r))_s ((a))_s ((u))_s \quad (-x^2yz)^s}{((k))_s ((w))_s ((c))_s ((v))_s (\alpha+1)_s\ s!}$$

$$F^{(2)}\left[\begin{matrix}(a)+s : (\ell)+s; (b); \\ (c)+s: (k)+s; (d);\end{matrix} -xz, t\right] F^{(2)}\left[\begin{matrix}(u)+s:(r)+s; (e); \\ (v)+s: (w)+s; (f);\end{matrix} xy, \omega\right], \qquad (4.5.2)$$

$$\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}((u))_{m+n} ((v))_{m+n}((h))_{m+n} \quad x^m (-x)^n}{((g))_{m+n}((e))_{m+n}((f))_{m+n} ((t))_{m+n} \quad m!\ n!}$$

$$F^{(2)}\left[\begin{matrix}(a) : 1-(e)-m-n, (v)+m+n, (q), -m; (b'); \\ (c) : 1-(u)-m-n, (f)+m+n, (p); (d');\end{matrix} (-1)^{1+E-U}y, \omega\right]$$

$$= F^{(2)}\left[\begin{matrix}(a) : (d), (v/2), (v/2)+ 1/2, (q), (h); (b'); \\ (c) : (g), (f/2), (f/2)+1/2, (p), (t); (d');\end{matrix} 4^{V-E}\, xy, \omega\right] \qquad (4.5.3)$$

Note that, when $y, \omega \to 0$, $x = \frac{x}{(1-x)}$, $L=L'+1$ and $\ell_{L'+1}=\alpha+1$, formula (4.5.2) would reduce to a result due to Srivastava [87,p.94(7.1)]

$$\sum_{m=0}^{\infty} \frac{(\alpha+1)_m}{m!} \quad F^{(2)}\left[\begin{matrix}(a) : (\ell), -m; (b); \\ (c) : (k) \quad ; (d);\end{matrix} z,t \quad x^m\right]$$

$$= (1-x)^{-(\alpha+1)} \quad F^{(2)}\left[\begin{matrix}(a) : (\ell), \alpha+1; (b); \\ (c) : (k) \quad ; (d);\end{matrix} -xz/(1-x)\ ,\ t\right]$$

Also for D=G=U=V=E=F=T=0, H=1, (4.5.3) reduces to the same result of Srivastava mentioned above. It may be of interest to remark that the relations (4.4.4) and (4.4.6) can also be specialized fairly easily to yield a large number of results involving series of the type :

$$\sum_{m,n=0}^{\infty} \frac{(\alpha+1)_{m+n}\, x^m(-x)^n}{m!\, n!} \phi(\omega,y)\, \phi^*(u,v) \tag{4.5.4}$$

$$\sum_{m,n=0}^{\infty} \frac{(\alpha+1)_{m+n}\, x^m(-x)^n}{m!\, n!} \Psi(\omega,y,z)\, \Psi^*(u,v,s) \tag{4.5.5}$$

$$\sum_{m,n=0}^{\infty} \frac{(\alpha+1)_{m+n}\, x^m(-x)^n}{m!\, n!} \Omega(x_1,\ldots,x_n)\, \Omega^*(y_1,\ldots,y_r) \tag{4.5.6}$$

$$\sum_{m,n=0}^{\infty} \frac{C(m+n)\, x^m(-x)^n}{m!\, n!} \phi(\omega,y) \tag{4.5.7}$$

$$\sum_{m,n=0}^{\infty} \frac{C(m+n)\, x^m(-x)^n}{m!\, n!} \Psi(\omega,y,z) \tag{4.5.8}$$

$$\sum_{m,n=0}^{\infty} \frac{C(m+n)\, x^m(-x)^n}{m!\, n!} \Omega(x_1,\ldots,x_n) \tag{4.5.9}$$

where $\phi(\omega,y)$ and $\phi^*(u,v)$ are one or the other of the Appell's functions F_1, F_2 and F_3, [94,p.53], $\Psi(\omega, y, z)$ and $\Psi^*(u,v,s)$ are hypergeometric functions of three variables $F^{(3)}$[94,p.69(39)], $\Omega(x_1,\ldots,x_n)$ and $\Omega^*(y_1,\ldots,y_r)$ are one or the other of the Lauricella's functions $F_A^{(n)}$, $F_B^{(n)}$ and $F_D^{(n)}$ [94, p. 60] and C(m+n) is function of (m+n) only and is independent of any variable.

For instance, in terms of Lauricella functions $F_A^{(n)}$ and $F_D^{(n)}$, equation (4.4.4) would give us the following special cases of type (4.5.6).

$$\sum_{m,n=0}^{\infty} \frac{(\alpha+1)_{m+n}\, x^m (-x)^n}{m!\, n!} F_A^{(n+1)}[a, b_1,\ldots\ldots,b_n, -m; d_1,\ldots.,d_n, k; z_1,\ldots,z_n,z]$$

$$F_A^{(r+1)}[u,e_1,\ldots\ldots\ldots, e_r, -n; f_1,\ldots.,f_r, w; y_1,\ldots, y_r, y]$$

$$=\sum_{s=0}^{\infty} \frac{(a)_s (u)_s (\alpha+1)_s(-x^2yz)^s}{(k)_s(w)_s\ s!} F_A^{(n+1)}[a+s,b_1,\ldots,b_n, 1+\alpha+s; d_1,\ldots,d_n, k+s;z_1,\ldots,z_n, -xz]$$

$$F_A^{(r+1)}[u+s, e_1,\ldots,e_r, 1+\alpha+s; f_1,\ldots,f_r, w+s; y_1,\ldots,y_r, xy] \tag{4.5.10}$$

$$\sum_{m,n=0}^{\infty} \frac{(\alpha+1)_{m+n}\, x^m (-x)^n}{m!\, n!} \qquad F_D^{(n+1)}[a, b_1,\ldots\ldots,b_n, -m; c; z_1,\ldots,z_n, z]$$

$$F_D^{(r+1)}[u, e_1,\ldots\ldots,e_r, -n; v; y_1,\ldots, y_r, y]$$

$$=\sum_{s=0}^{\infty} \frac{(a)_s (u)_s (\alpha+1)_s(-x^2yz)^s}{(c)_s(v)_s\ \ s!} F_D^{(n+1)}[a+s,b_1,\ldots,b_n, 1+\alpha+s; c+s; z_1,\ldots,z_n, -xz]$$

$$F_D^{(r+1)}[u+s, e_1,\ldots,e_r, 1+\alpha+s; v+s; y_1,\ldots,y_r, xy] \tag{4.5.11}$$

If in (4.5.10), we set $y_1=\ldots.=y_r=0$, and u=w, we obtain

$$[1+x(1-y)]^{-\alpha-1} \sum_{m=0}^{\infty} \frac{(\alpha+1)_m}{m!} \left[\frac{x}{1+x(1-y)}\right]^m F_A^{(n+1)}[a,b_1,\ldots, b_n, -m; d_1,\ldots.,d_n, k ;z_1,\ldots,z_n,z]$$

$$= (1-xy)^{-\alpha-1} \sum_{s=0}^{\infty} \frac{(a)_s(\alpha+1)_s}{(k)_s\ s!} \left[\frac{-x^2yz}{(1-xy)}\right]^s$$

$$F_A^{(n+1)}[a+s, b_1,\ldots\ldots,b_n, 1+\alpha+s; d_1,\ldots,d_n, k+s; z_1,\ldots,z_n, -xz] \tag{4.5.12}$$

For y=1, (4.5.12) reduces to a multivariable extension of a known result [94,p.293(12)], (see (4.3.1)).

On taking $z_2=\ldots.=z_n=y_1=\ldots=y_r=y=0$, (4.5.10) reduces to

$$\sum_{m=0}^{\infty} \frac{(\alpha+1)_m x^m}{m!} F_2 [a, -m, b; k, d; z, \omega]$$

$$= (1-x)^{-(\alpha+1)} F_2 \left[a, 1+\alpha, b; k, d; \frac{-xz}{(1-x)}, \omega\right] \tag{4.5.13}$$

Now, on replacing $\alpha+1$ and x by λ and x/λ in (4.5.13) respectively, letting $\lambda\to\infty$ and using the results [87,p.94(7.3)]

$$\lim_{\lambda\to\infty} (1-z/\lambda)^{-\lambda}=e^z \qquad \text{and} \lim_{|\lambda|\to\infty} (\lambda)_n (z/\lambda)^n= z^n \tag{4.5.14}$$

equation (4.5.13) reduces to a known result of Srivastava [87,p.94(7.5)]

$$\sum_{m=0}^{\infty} F_2 [a, -m, b; k, d; z, \omega] \frac{x^m}{m!}$$

$$= e^x \; \Psi_1 [a, b; k, d ; -xz , \omega]$$

where Ψ_1 is Humbert function defined by (1.5.3). On setting $z_1=\ldots.=z_n=z$, $y_1=\ldots..= y_r=y$ and using the reduction formula [93, p.34(6)]

$$F_D^{(n)}[(a,b_1,\ldots.,b_n; c; x,\ldots, x] = {}_2F_1 [a,b_1+\ldots.+b_n; c; x]$$

equation (4.5.11) reduces to

$$\sum_{m,n=0}^{\infty} \frac{(\alpha+1)_{m+n} x^m (-x)^n}{m!\ n!} {}_2F_1 \left[\begin{matrix} b_1+\ldots+b_n-m, a; \\ c; \end{matrix} z\right] {}_2F_1 \left[\begin{matrix} e_1+\ldots+e_r,-n,u; \\ v; \end{matrix} y\right]$$

$$= \sum_{s=0}^{\infty} \frac{(a)_s(u)_s(\alpha+1)_s(-x^2yz)^s}{(c)_s (v)_s s!} F_1 [a+s,1+\alpha+s, b_1+\ldots.+b_n ; c+s; -xz, z]$$

$$F_1 [u+s, 1+\alpha+s, e_1+\ldots+e_r ; v+s; xy, y] \tag{4.5.15}$$

On putting s, $y \to 0$, $b_i=0$, $i=2,3,\ldots,n$, (4.5.15) evidently reduces to a known result [94,p.150(44)]

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} {}_2F_1\,[b-n,\ a;\ c;\ z]\ x^n = (1-x)^{-\lambda}\ F_1\,[a,\ b,\ \lambda;\ c;\ z,\ zx/(x-1)], \quad (4.5.16)$$

$$|x| < 1.$$

For $b_i=0$, $i=2,3,\ldots,n$ and $u=v$, formula (4.5.15) may at once written in the form

$$= \sum_{m=0}^{\infty} \frac{(\alpha+1)_m}{m!} \left[\frac{x}{1+x(1-y)}\right]^m {}_2F_1\left[\begin{matrix} b-m,\ a; \\ c; \end{matrix}\ z\right] = (1-xy)^{-\alpha-1}\,(1+x(1-y))^{\alpha+1}$$

$$F^{(3)}\left[\begin{matrix} a::\alpha+1;\ -;\ -:\ -;\ -;\ b; \\ c::\ -;\ -;\ -:\ -;\ -;-; \end{matrix}\ -xz,\ \frac{-x^2yz}{(1-xy)},\ z\right] \quad (4.5.17)$$

where $F^{(3)}[x,y,z]$ is Srivastava's triple hypergeometric series [94,p.69(39) and (40)] defined by (1.8.1). It is important to note that, the left-hand side of equation (4.5.17) can be summed by using one or other of the results [94,p.150(43),(44) and 151(45)], to obtain some transformation formulae for $F^{(3)}$ (right-hand side of (4.5.17)) in the form of functions of Gaussian ${}_2F_1$, Appell F_1 and a special case of Kampe' de Fe'riet of two variables $F^{(2)}$. For example, if in (4.5.17) we set $\alpha+1=\lambda$, $t=x/(1+x(1-y))$ and use (4.5.16) we get the transformation formula

$$F_1[a,b,\lambda;\ c;\ z,zx/(xy-1)] = F^{(3)}\left[\begin{matrix} a::\ \lambda;\ -;\ -:\ -;\ -;\ b; \\ c::\ -;\ -;\ -:\ -;\ -;-; \end{matrix}\ -xz,\ \frac{-x^2yz}{(1-xy)},\ z\right]$$

(4.5.18)

Also a similar transformations can be obtained from the main result (4.4.4). Further, on setting $y_1=...=y_r=y=z_2=...=z_n=0$, in (4.5.11), replacing $\alpha+1$ and x by λ and x/λ respectively, letting $\lambda\to\infty$ and using (4.5.14), formula (4.5.11) reduces to another known result [87, p.94 (7.4)].

$$\sum_{m=0}^{\infty} F_1 [a, b, -m; k; z, \omega] \frac{x^m}{m!}$$

$$= e^x \; \Phi_1 [a, b; k; -xz, \omega] \tag{4.5.19}$$

where Φ_1 is Humbert function of two variables defined by (1.5.1).

On other hand, as particular cases of our result (4.5.3) we obtain three linear generating relations involving the Appell functions F_2 and F_3. Indeed we have the following generating relations of the type (4.5.7).

$$\sum_{m,n=0}^{\infty} \frac{(u)_{m+n}(h)_{m+n} x^m (-x)^n}{(t)_{m+n}\, m!\, n!} F_2 [a, -m, b; 1-u-m-n, d; y, \omega]$$

$$= F_2 [a, h, b; t, d; xy, \omega] \tag{4.5.20}$$

$$\sum_{m,n=0}^{\infty} \frac{(d)_{m+n}(h)_{m+n} x^m (-x)^n}{(e)_{m+n}\, m!\, n!} F_3 [-m, b_1, 1-e-m-n, b_2; c; y, \omega]$$

$$= F_3 [d, b_1, h, b_2; c; xy/4, \omega] \tag{4.5.21}$$

$$\sum_{m,n=0}^{\infty} \frac{(v)_{m+n}\, x^m (-x)^n}{m!\, n!} F_3 [-m, b_1, v+m+n, b_2; c; y, \omega]$$

$$= F_3 [v/2, b_1, v/2+1/2, b_2; c; -4xy, \omega] \tag{4.5.22}$$

Finally on setting $H=B^{(i)}=1$, $D^{(i)}=0, i=1,\ldots, n$ and $E=F=V=U=T=Q=P=D=G=0$ in (4.4.6) together with the reduction formula for the generalized Kampe' de Fe'riet series of several variables [93,p.39 (32)], we get a generalization of (4.3.4) given by

$$\sum_{m=0}^{\infty} \frac{(h)_m}{m!} \; {}_{A+1}F_B \left[\begin{matrix} b_1+\ldots\ldots+b_n-m, \; (a); \\ (c); \end{matrix} \; y \right] x^m$$

$$= (1-x)^{-h} \, F^{(2)} \left[\begin{matrix} (a):b_1+\ldots+b_n;h; \\ (c): \text{———} ;\text{—}; \end{matrix} \; y \, , \, \frac{xy}{(x-1)} \right], \; |x| < 1. \qquad (4.5.23)$$

Now, if in (4.5.23), we set $b_i=0$, $i=2,3,\ldots,n$, then it reduces to a known result of Srivastava [94,p.150 (43)].

$$\sum_{m=0}^{\infty} \frac{(h)_m}{m!} \; {}_{A+1}F_B \left[\begin{matrix} b-m, \; (a); \\ (c); \end{matrix} \; y \right] x^m$$

$$= (1-x)^{-h} \, F^{(2)} \left[\begin{matrix} (a): b \;;\; h; \\ (c): \text{—} \;;\text{—}; \end{matrix} \; y \, , \, \frac{xy}{(x-1)} \right], \; |x| < 1. \qquad (4.5.24)$$

CHAPTER-V

ON GENERALIZATION OF GENERATING FUNCTIONS INVOLVING BESSEL AND LAGUERRE POLYNOMIALS

5.1 INTRODUCTION

In the theory of special functions, the Bessel and Laguerre polynomials play the same role as in a number of other branches of mathematics and physics. For example, the Bessel polynomials arise in the theory of electromagnetism and in the study of free vibrations of a circular membrance [55]. On other hand, it is well known that, Laguerre polynomials occur in problems involving the integration of Helmholtz's equation in parabolic coordinates, in the theory of the hydrogen atom etc. [57].

The purpose of this chapter is to establish Bessel polynomials in several variables, which provide multivariable generalization of known Bessel polynomials. Certain integral representations have been obtained for these multivariable Bessel polynomials. In section 5.3 our main generating functions for $y^{(\alpha_1,\ldots,\alpha_n;\beta)}_{m_1,\ldots,m_n}(x_1,\ldots\ldots,x_n)$ which are linear and partly bilateral and partly unilateral are obtained with the help of a result of Exton [31].

Section 5.4 deals with a technique of integral operators for obtaining generating functions of Lauricella function $F_A^{(n)}$ and Horn function ${}^{(k)}H_4^{(n)}$ which are partly bilateral and partly unilateral.

The generalized Horn's function ${}^{(k)}H_4^{(n)}$ of Exton [27] is a function which not only generalizes Horn's functions H_4, $H_4^{(p)}$ [48] but also Lauricella's $F_A^{(n)}$, $F_C^{(n)}$, Appell's F_2, F_4 and ${}_2F_1$.

Many known results of Al-Salam [3], Agrawal [1], Mumtaz and Khursheed [63], Pathan and Kamarujjama [65] and Pathan and Yasmeen [69] are shown as special cases of the results of this chapter.

5.2 DEFINITION AND INTEGRAL REPRESENTATIONS OF THE BESSEL POLYNOMIALS $y_{m_1,\ldots,m_n}^{(\alpha_1,\ldots,\alpha_n;\beta)}(x_1,\ldots,x_n)$

Exton [32,p.4(3.1)] has introduced a Bessel polynomial in several variables. This is defined as follows :

$$y_{m_1,\ldots,m_n}(x_1,\ldots,x_n;a)=\sum_{k_1=0}^{m_1}\cdots\cdots\sum_{k_n=0}^{m_n}(a-1+m_1+\ldots\ldots m_n)_{k_1+\ldots+k_n}(-m)_{k_1}\ldots\ldots(-m)_{k_n}$$

$$\frac{(-x_1)^{k_1}}{k_1!}\cdots\cdots\frac{(-x_n)^{k_n}}{k_n!}\,. \qquad (5.2.1)$$

If all but one of the variables are suppressed, we recover the Bessel polynomial

$$y_m(a;x) = {}_2F_0\,[-m,\ a-1+m;..;\ -x]\ , \qquad (5.2.2)$$

which on replacing x by x/b, gives us the Bessel polynomials

$$y_m(a,b;x) = {}_2F_0\,[-m,\ a-1+m;-;\ -x/b]\ . \qquad (5.2.3)$$

The Bessel polynomials (5.2.3) were introduced by Krall and Frink [53] in connection with solution of the wave equation in spherical coordinates.

Several other authors including Agarwal [1], Carlitz [11], Grosswald [37], Al-Salam [3], Chatterjea ([14] and [15]) and Mumtaz and Khursheed [63] have contributed to the study of Bessel polynomials.

The polynomial $y^{(\alpha_1,\ldots,\alpha_n;\,\beta)}_{m_1,\ldots,m_n}(x_1,\ldots,x_n)$ is defined as follows :

$$y^{(\alpha_1,\ldots,\alpha_n;\,\beta)}_{m_1,\ldots,m_n}(x_1,\ldots,x_n) = \sum_{k_1=0}^{m_1}\cdots\cdots\sum_{k_n=0}^{m_n} (1+\beta+\alpha_1 m_1+\ldots+\alpha_n m_n)_{k_1+\ldots\ldots+k_n}$$

$$\prod_{j=1}^{n} \left\{\binom{m_j}{k_j} z_j^{k_j}\right\} . \tag{5.2.4}$$

If in (5.2.4), we set $\alpha_j=1$, $(j=1,2,\ldots,n)$ and $\beta=a-2$, we shall readily obtain Exton's Bessel polynomial (5.2.1). On setting n=1 and replacing x by x/2, (5.2.4) reduces to

$$y_m^{(\alpha,\beta)}(x) = {}_2F_0\,[-m,\ \alpha m+\beta+1;\ -;\ -x/2] , \tag{5.2.5}$$

a polynomial introduced by Mumtaz and Khursheed [63, p.152(2.1)].

Also, Bessel polynomials due to Al-Salam [3,p.529(2.2)], Chatterjea [5] and Krall and Frink (5.2.3) are contained in (5.2.4).

Erde'lyi [21] defined the multivariable Laguerre polynomials by the relation

$$L^{(\alpha)}_{m_1,\ldots,m_n}(x_1,\ldots,x_n) = \frac{(\alpha+1)_{m_1+\ldots+m_n}}{m_1!\ldots..m_n!}\ \Phi_2^{(n)}\,[-m_1,\ldots,-m_n;\ \alpha+1;\ x_1,\ldots,x_n], \tag{5.2.6}$$

where $\Phi_2^{(n)}$ is a confluent hypergeometric function of n–variables [94, p.62(10)] defined by (1.10.6).

In view of the identities (cf. (1.1.13) and (1.1.15))

$$(-n)_k = \frac{(-1)^k\ n!}{(n-k)!} \quad , \qquad (\lambda)_{n-k} = \frac{(-1)^k\ (\lambda)_n}{(1-\lambda-n)_k} , \tag{5.2.7}$$

one can rewrite (2.3) in the form

$$L^{(\alpha)}_{m_1,\ldots,m_n}(x_1,\ldots,x_n) = \prod_{j=1}^{n}\left\{\frac{-x_j^{m_j}}{m_j}\right\} \sum_{k_1=0}^{m_1}\cdots\cdots\sum_{k_n=0}^{m_n}(-\alpha-m_1-\ldots-m_n)_{k_1+\ldots+k_n}$$

$$\prod_{j=1}^{n}\left\{\binom{m_j}{k_j}\left(\frac{1}{z_j}\right)^{k_j}\right\}. \tag{5.2.8}$$

From (5.2.4) and (5.2.8), we see that the Bessel polynomials are essentially Laguerre polynomials. In fact we have

$$y^{(\alpha_1,\ldots,\alpha_n;\beta)}_{m_1,\ldots,m_n}(x_1,\ldots,x_n) = \prod_{j=1}^{n}\left\{m_j!\,(-x_j)^{m_j}\right\} L^{(-1-\beta-m_1-\ldots-m_n-v)}_{m_1,\ldots,m_n}(1/x_1,\ldots,1/x_n), \tag{5.2.9}$$

where throughout this chapter

$$v = \alpha_1 m_1+\ldots+\alpha_n m_n \quad , \; m_j \; , \; (j=1,2,\ldots n) \text{ are integers.} \tag{5.2.10}$$

For n=1, α=1 and x replaced by x/2, (5.2.9) reduces to

$$y^{(\beta)}_m(x) = m!(-x/2)^m\, L^{(-2m-\beta-1)}_m(2/x)\;, \tag{5.2.11}$$

a result due to Al-Salam [3,p.530(2.5)].

Now we will present a number of integral representations for the generalized Bessel polynomials of several variables (5.2.4) in terms of Euler and Laplace integrals.

Indeed, it is easy to derive the following integral representations

$$y^{(\alpha_1,\ldots,\alpha_n;\beta)}_{m_1,\ldots,m_n}(x_1,\ldots,x_n)= \frac{s^{v+\beta+1}}{\Gamma(v+\beta+1)}\int_0^{\infty} t^{v+\beta}\, e^{-st}\prod_{j=1}^{n}\left\{(1+x_j ts)^{m_j}\right\}dt, \tag{5.2.12}$$

$$F^{2:1;\ldots;1}_{1:0;\ldots;0}\left[\begin{array}{l} 1+\beta+v\;,\;\lambda\;:\;-m_1;\ldots\ldots;-m_n; \\ \qquad\qquad\qquad\qquad\qquad -x_1,\ldots\ldots,-x_n \\ \lambda+\mu \qquad\quad :-;\ldots\ldots;-; \end{array}\right]$$

$$= \frac{\Gamma(\lambda+\mu)}{\Gamma\lambda\,\Gamma\mu} \int_0^1 t^{\lambda-1} (1-t)^{\mu-1}\, y^{(\alpha_1,\ldots,\alpha_n;\,\beta)}_{m_1,\ldots,m_n} (x_1 t,\ldots, x_n t)\, dt\ , \tag{5.2.13}$$

$$F^{2:1;\ldots;1}_{1:0;\ldots;0} \left[\begin{matrix} 1+\beta+v\ ,\ \mu: -m_1;\ldots\ldots\ldots;-m_n; \\ \qquad\qquad\qquad\qquad -x_1,\ldots\ldots\ldots,-x_n \\ \lambda+\mu \quad :-;\ldots\ldots\ldots;-; \end{matrix} \right]$$

$$= \frac{\Gamma(\lambda+\mu)}{\Gamma\lambda\,\Gamma\mu} \int_0^1 t^{\lambda-1} (1-t)^{\mu-1}\, y^{(\alpha_1,\ldots,\alpha_n;\beta)}_{m_1,\ldots,m_n} (\, x_1(1-t),\ldots,x_n(1-t))dt, \tag{5.2.14}$$

where $F^{A:B';\ldots;\,B^{(n)}}_{C:D';\ldots;\,D^{(n)}} [x_1,\ldots,x_n]$ is the generalized Kampe' de Fe'riet function of n-variables defined by (1.11.2).

$$\frac{-\pi\ \sin\pi(u+u'+\beta+\gamma)\ (u+u'+\beta+\gamma+1)}{\sin\pi(u'+\gamma)\ \sin\pi(u+\beta)\ (u'+\gamma+1)\ (u+\beta+1)}\ y^{(\alpha\ ;\beta+\gamma)}_{m_1+k_1+\ldots+m_n+k_n} (x)$$

$$= \int_0^1 y^{(\alpha,\ldots,\alpha;\beta)}_{m_1,\ldots m_n} \left(\frac{x}{t},\ \ldots,\frac{x}{t}\right) y^{(\alpha,\ldots,\alpha;\gamma)}_{k_1,\ldots,k_n} \left(\frac{x}{1-t},\ldots,\ \frac{x}{1-t}\right) t^{-(u+\beta+1)}\ (1-t)^{-(u'+\gamma+1)}\ dt\ , \tag{5.2.15}$$

where $u = \alpha(m_1+\ldots+m_n)$ and $u'=\alpha(k_1+\ldots+k_n)$.

<u>Proofs of equations (5.2.12) to (5.2.15)</u> : To prove (5.2.12), denote, for convenience, the right-hand side by V, then it is easily seen that

$$V=\frac{s^{v+\beta+k_1+\ldots+k_n+1}}{\Gamma\, v+\beta+1} \sum_{k_1,\ldots,k_n=0}^{\infty} \prod_{j=1}^{n} \left\{ \frac{-x_j^{k_j}}{k_j\,!}\ (-m_j)_{k_j} \right\}$$
$$\int_0^{\infty} t^{(v+\beta+k_1+\ldots\ldots+k_n+1)-1}\ e^{-st}\ dt\ .$$

Now, evaluating the integral and using (5.2.4), we arrive at the result (5.2.12). The proofs of equation (5.2.13) to (5.2.15) are similar to that of equation (5.2.12).

Some special cases of the results mentioned above are worthy of note. Indeed upon setting n=1 in (5.2.12), if we let s=1, α=1, β=a–2 and replace x by x/b, (5.2.12) readily yields

$$y_n\,(a,b,x) = \frac{1}{\Gamma(\alpha+n-1)} \int_0^{\infty} t^{a-2+n} \left(1+\frac{xt}{b}\right)^n e^{-t}\, dt\,, \tag{5.2.16}$$

a result due to Agarwal [1]. On putting $\mu=\beta$, λ=v+1 in (5.2.13), we get

$$y_{m_1,\dots,m_n}^{(\alpha_1,\dots,\alpha_n;\,0)}(x_1,\dots,x_n)= \frac{\Gamma(v+\beta+1)}{\Gamma(v+1)\,\Gamma\beta}\int_0^1 t^v\,(1-t)^{\beta-1}\; y_{m_1,\dots,m_n}^{(\alpha_1,\dots,\alpha_n;\,\beta)}(x_1t,\dots,x_nt)dt. \tag{5.2.17}$$

On other hand, setting $\lambda=m_1+\dots+m_n+\beta+1$, $\mu=v-m_1-\dots-m_n$, in (5.2.13), we obtain

$$y_{m_1,\dots,m_n}^{(1,\dots,1;\,\beta)}(x_1,\dots,x_n)=\frac{\Gamma(v+\beta+1)}{\Gamma(m_1+\dots+m_n+\beta+1)\,\Gamma(v-m_1-\dots-m_n)}\int_0^1 t^{m_1+\dots+m_n+\beta}\;(1-t)^{v-m_1-\dots-m_n-1}$$

$$y_{m_1,\dots,m_n}^{(\alpha_1,\dots,\alpha_n;\beta)}(x_1t,\dots,x_nt)\,. \tag{5.2.18}$$

On replacing λ and μ in (5.2.13) by $v+\beta+\lambda+1$ and $-\lambda$ respectively, equation (5.2.13) reduces to

$$y_{m_1,\dots,m_n}^{(\alpha_1,\dots,\alpha_n;\beta+\lambda)}(x_1,\dots,x_n) = \frac{-\sin\pi(\lambda)\,\Gamma(1+\lambda)\,\Gamma(v+\beta+1)}{\pi\,\Gamma(v+\beta+\lambda+1)}\int_0^1 t^{v+\beta+\lambda}\;(1-t)^{-(\lambda+1)}$$

$$y_{m_1,\dots,m_n}^{(\alpha_1,\dots,\alpha_n;\,\beta)}(x_1t,\dots,x_nt)\;\;dt\,. \tag{5.2.19}$$

Further, if in (5.2.14), we replace λ and μ by μ and $v+\beta-\mu+1$ respectively, (5.2.14) yields

$$y_{m_1,\ldots,m_n}^{(\alpha_1,\ldots,\alpha_n;\beta-\mu)}(x_1,\ldots,x_n) = \frac{\Gamma(v+\beta+1)}{\Gamma\mu\,\Gamma(v+\beta-\mu+1)} \int_0^1 t^{\mu-1}(1-t)^{v+\beta-\mu}$$

$$y_{m_1,\ldots,m_n}^{(\alpha_1,\ldots,\alpha_n;\beta)}\big(x_1(1-t),\ldots,x_n(1-t)\big)\ dt\ . \qquad (5.2.20)$$

For $\mu=1$, $\lambda=(v+\beta)$, (5.2.13) reduces to

$$y_{m_1,\ldots,m_n}^{(\alpha_1,\ldots,\alpha_n;\beta-1)}(x_1,\ldots,x_n) = (v+\beta)\int_0^1 t^{v+\beta-1}\ y_{m_1,\ldots,m_n}^{(\alpha_1,\ldots,\alpha_n;\beta)}(x_1t,\ldots,x_nt)\ dt\ . \qquad (5.2.21)$$

Finally, if in (5.2.13), we put $\lambda=1$ and $\mu=v+\beta$, (5.2.14) reduces to

$$y_{m_1,\ldots,m_n}^{(\alpha_1,\ldots,\alpha_n;\beta-1)}(x_1,\ldots,x_n) = (v+\beta)\int_0^1 (1-t)^{v+\beta-1}\ y_{m_1,\ldots,m_n}^{(\alpha_1,\ldots,\alpha_n;\beta)}\big(x_1(1-t),\ldots\ldots,x_n(1-t)\big)dt\ . \qquad (5.2.22)$$

Integral representations (5.2.12) to (5.2.15), (5.2.17) to (5.2.19), (5.2.21) and (5.2.22) are generalizations of known results obtained by Mumtaz and Khursheed [63, Equations (3.5), (3.3) to (3.4), (3.6) to (3.9), (3.14) and (3.15)] respectively.

5.3 GENERATING FUNCTIONS FOR THE BESSEL POLYNOMIALS $y_{m_1,\ldots,m_n}^{(\alpha_1,\ldots,\alpha_n;\beta)}(x_1,\ldots,x_n)$

It is not difficult to derive the following basic generating relations:

$$e^{t_1+\ldots+t_n}(1-x_1t-\ldots-x_nt_n)^{-(\beta+1)} = \sum_{m_1,\ldots,m_n=0}^{\infty} y_{m_1,\ldots,m_n}^{(\alpha_1,\ldots,\alpha_n;\,\beta-v)}(x_1,\ldots,x_n)\,\frac{t_1^{m_1}}{m_1!}\cdots\frac{t_n^{m_n}}{m_n!}, \qquad (5.3.1)$$

$$e^{t_1+\dots+t_n}(1-x_1t_1-\dots-x_nt_n)^{-(1+\beta+\alpha_1+\dots+\alpha_n)}$$

$$=\sum_{m_1,\dots,m_n=0}^{\infty} y_{m_1,\dots,m_n}^{(\alpha_1/m_1,\dots,\alpha_n/m_n;\ \beta)}(x_1,\dots,x_n)\frac{t_1^{m_1}}{m_1!}\cdots\frac{t_n^{m_n}}{m_n!}. \qquad (5.3.2)$$

<u>**Proof of equation (5.3.1) and (5.3.2)**</u> : To prove (5.3.1), we write the left-hand side in the form :

$$V=\sum_{s_1,\dots,s_n,k_1,\dots,k_n=0}^{\infty}(\beta+1)_{k_1+\dots+k_n}\prod_{j=1}^{n}\left\{\frac{-z_j^{k_j}\,t_j^{s_j+k_j}}{k_j!\ s_j!}\right\}.$$

Replace $s_1+k_1,\dots,$ and s_n+k_n by $m_1,\dots,$ and m_n respectively. Then after rearrangement justified by absolute convergence of the above series and using the definition (5.2.4), we arrive at (5.3.1). The proof of (5.3.2) is similar to that of (5.3.1).

In formula (5.2.12), put s=1, v+β=k, where v is defined by (5.2.10) and k is integer, multiply throughout by $(-\lambda)^k$ and then sum to get

$$y_{m_1,\dots,m_n}^{(\alpha_1,\dots,\alpha_n;\ -v)}\left(\frac{x_1}{1+\lambda},\dots,\frac{x_n}{1+\lambda}\right)=(1+\lambda)\sum_{k=0}^{\infty}y_{(m_1,\dots,m_n}^{(\alpha_1,\dots,\alpha_n;\ k-v)}(x_1,\dots,x_n)(-\lambda)^k. \qquad (5.3.3)$$

Similarly, we find that

$$\sum_{k=0}^{\infty}y_{m_1,\dots,m_n}^{(\alpha_1,\dots,\alpha_n;\ k-v)}(x_1,\dots,x_n)\frac{(-\lambda)^k}{k!}=\int_0^{\infty}e^{-t}\prod_{j=1}^{n}\left\{(1+x_jt)^{m_j}\right\}J_0(2\sqrt{\lambda t})\,dt. \qquad (5.3.4)$$

Evaluating the right-hand side, we get

$$\sum_{k=0}^{\infty}y_{m_1,\dots,m_n}^{(\alpha_1,\dots,\alpha_n;\ k-v)}(x_1,\dots,x_n)\frac{(-\lambda)^k}{k!}=e^{-\lambda}\sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty}\prod_{j=1}^{n}\left\{\frac{(-m_j)_{k_j}(-x_j)^{k_j}}{k_j!}\right\}$$

$$\overline{|(k_1+\dots+k_n+1)}\ L_{k_1+\dots+k_n}(\lambda), \qquad (5.3.5)$$

where $L_n(x)$ is Laguerre polynomial [73,p.200(2)].

On setting n=1, α=1 in (5.3.5) and replacing x by x/2, it reduces to a known result due to Al-Salam [3, p.536 (6.5)]. In the same manner one can derive the following formulae :

$$\sum_{k=0}^{\infty} y_{m_1,\ldots,m_n}^{(\alpha_1,\ldots,\alpha_n;\, 2k-v)}(x_1,\ldots,x_n)\,(-\lambda^2)^k = \int_0^{\infty} e^{-t} \prod_{j=1}^{n} \{(1+x_j t)^{m_j}\} \cos(\lambda t)\, dt, \quad (5.3.6)$$

$$\sum_{k=0}^{\infty} y_{m_1,\ldots,m_n}^{(\alpha_1,\ldots,\alpha_n;\, 2k-v+1)}(x_1,\ldots,x_n)\,(-\lambda^2)^k = (1/\lambda)\int_0^{\infty} e^{-t} \prod_{j=1}^{n} \{(1+x_j t)^{m_j}\} \sin(\lambda t)\, dt, \quad (5.3.7)$$

$$\sum_{k_1,\ldots,k_n=0}^{\infty} y_{m_1,\ldots,m_n}^{(\alpha_1,\ldots,\alpha_n;\, \beta-k_1-\ldots-k_n)}(x_1,\ldots,x_n) \prod_{j=1}^{n} \left\{\frac{u_j^{k_j}}{k_j}\right\} = \frac{1}{\Gamma(\beta+v+1)} \int_0^{\infty} e^{-t} \prod_{j=1}^{n} \{(1+x_j t)^{m_j}\}$$

$$(t+u_1+\ldots+u_n)^{v+\beta}\, dt\,. \quad (5.3.8)$$

Now, by using the definition of the generalized Bessel polynomials (5.2.4), it is easy to derive the following results :

$$\sum_{m_1,\ldots,m_n=0}^{\infty} y_{m_1,\ldots,m_n}^{(\alpha_1,\ldots,\alpha_n;\beta-v)}(x_1,\ldots,x_n) \prod_{j=1}^{n} \{t_j^{m_j}\} \cong \prod_{j=1}^{n} \{ (1+t_j)^{-1} \}$$

$$F^{1:1;\ldots;1}_{0:0;\ldots;0}\left[\begin{matrix} \beta+1:1;\ldots\ldots;1; \\ \text{——}:-;:\ldots\ldots;-; \end{matrix}\ \frac{x_1 t_1}{(1-t_1)},\ldots,\frac{x_n t}{(1-t_n)}\right], \quad (5.3.9)$$

$$\sum_{m_1,\ldots,m_n=0}^{\infty} y_{m_1,\ldots,m_n}^{(\alpha_1/m_1,\ldots,\alpha_n/m_n;\, \beta)}(x_1,\ldots,x_n) \prod_{j=1}^{n} \{t_j^{m_j}\} \cong \prod_{j=1}^{n} \{(1+t_j)^{-1}\}$$

$$F^{1:1;\ldots;1}_{0:0;\ldots;0}\left[\begin{matrix} \alpha_1+\ldots+\alpha_n+\beta+1:1;\ldots\ldots;1; \\ \text{————}:-;:\ldots\ldots;-; \end{matrix}\ \frac{x_1 t_1}{(1-t_1)},\ldots,\frac{x_n t_n}{(1-t_n)}\right]. \quad (5.3.10)$$

The formulae (5.3.1) to (5.3.4) and (5.3.6) to (5.3.10) are generalizations of known results [63, Equations (4.1), (4.3), (4.6) to (4.10) and (4.4) to (4.5)] respectively.

Now, we derive a partly bilateral and partly unilateral generating relatgion for $y^{(\alpha_1,\dots,\alpha_n;\beta)}_{m_1,\dots,m_n}(x_1,\dots,x_n)$. We begin with the modified result of Exton [31], (cf. 2.1.3)).

$$\exp(s+t-zt/s) = \sum_{m=-\infty}^{\infty}\sum_{p=m^*}^{\infty} \frac{s^m\, t^p}{(m+p)!}\, L_p^{(m)}(z) \,. \tag{5.3.11}$$

On putting s=(1–s–t+zt/s) in (5.2.12) and making use of (5.3.11), we find that :

$$(1-s-t+zt/s)^{-(v+\beta+1)}\, y^{(\alpha_1,\dots,\alpha_n;\beta)}_{m_1,\dots\dots,m_n}\left(\frac{x_1}{1-s-t+zt/s},\dots\dots,\frac{x_n}{1-s-t+zt/s}\right)$$

$$= \frac{1}{\Gamma(v+\beta+1)} \sum_{m=-\infty}^{\infty}\sum_{p=m^*}^{\infty} \frac{s^m\, t^p}{m!\;p!} \int_0^\infty u^{(v+\beta+m+l+1)}\, e^{-u} \prod_{j=1}^{n} \left\{1+x_j u)^{m_j}\right\}$$

$$L_p^{(m)}(zu)\, du \,. \tag{5.3.12}$$

Evaluating the right-hand side, we get

$$(1-s-t+zt/s)^{-(v+\beta+1)}\, y^{(\alpha_1,\dots,\alpha_n;\beta)}_{m_1,\dots\dots,m_n}\left(\frac{x_1}{1-s-t+zt/s},\dots\dots,\frac{x_n}{1-s-t+zt/s}\right)$$

$$= \sum_{m=\infty}^{\infty}\sum_{p=m^*}^{\infty} \frac{s^m\, t^p}{m!\;p!}\,(v+\beta+1)_{m+p}\, F^{1:1;1;\dots;1}_{0:1;0;\dots;0}\left[\begin{matrix} v+\beta+m+p+1:\ -p;-m_1;\dots;m_n; \\ \text{———— } :m+1;\dots\dots;-; \end{matrix}\ z,-x_1,\dots,-x_n\right]. \tag{5.3.13}$$

Equation (5.3.13) gives a number of generating relations as special cases. We present some interesting spacial cases here. If in (5.3.13), we set n=1, replace x by x/2, it yields generating relation involving the Bessel polynomials $y_n^{(\alpha,\beta)}$ (cf. (5.2.5) :

$$(1-s-t+zt/s)^{-(\alpha n+\beta+1)}\ y_n^{(\alpha,\beta)}\left(\frac{x}{2(1-s-t+zt/s)}\right)$$

$$= \sum_{m=\infty}^{\infty}\sum_{p=m^*}^{\infty}\frac{s^m\, t^p}{m!\; p!}(\alpha n+\beta+1)_{m+p}\ F^{1:1;1}_{0:1;0}\left[\begin{array}{l}\alpha n+\beta+m+p+1:\ -p\ ;-n;\\ \overline{\qquad\qquad}\ :m+1;-;\end{array}\ z,\ -x/2\right], \qquad (5.3.14)$$

which for β=a-2, α=1 and x replaced by 2x/b, reduces to

$$(1-s-t+zt/s)^{-(a+n-1)}\ \mathcal{Y}_n\left(a,b;\ \frac{x}{1-s-t+zt/s}\right)$$

$$= \sum_{m=\infty}^{\infty}\sum_{p=m^*}^{\infty}\frac{s^m\, t^p}{m!\; p!}(a+n-1)_{m+p}\ F^{1:1;1}_{0:1;0}\left[\begin{array}{l}a+n+m+p-1:\ -p\ \ ;n;\\ \overline{\qquad\qquad}\ :m+1;-;\end{array}\ z,\ -x/b\right]. \qquad (5.3.15)$$

Next, for z→0, equations (5.3.13) to (5.3.15) reduce to the following elegant results :

$$(1-s-t)^{-(v+\beta+1)}\qquad y^{(\alpha_1,\ldots,\alpha_n;\beta)}_{m_1,\ldots,m_n}\left(\frac{x_1}{1-s-t},\ldots\ldots,\frac{x_n}{1-s-t}\right)$$

$$= \sum_{m=\infty}^{\infty}\sum_{p=m^*}^{\infty}\frac{s^m\, t^p}{m!\; p!}(v+\beta+1)_{m+p}\qquad y^{(\alpha_1,\ldots,\alpha_n;\beta+m+p)}_{m_1,\ldots,m_n}(x_1,\ldots,x_n), \qquad (5.3.16)$$

$$(1-s-t)^{-(\alpha n+\beta+1)}\qquad y_n^{(\alpha,\beta)}\left(\frac{x}{2(1-s-t)}\right)$$

$$= \sum_{m=\infty}^{\infty}\sum_{p=m^*}^{\infty}\frac{s^m\, t^p}{m!\; p!}(\alpha n+\beta+1)_{m+p}\ y_n^{(\alpha,\,\beta+m+p)}(x) \qquad (5.3.17)$$

and

$$(1-s-t)^{-(a+n-1)} \quad y_n \left(a,b; \frac{x}{1-s-t}\right)$$

$$= \sum_{m=-\infty}^{\infty} \sum_{p=m^*}^{\infty} \frac{s^m t^p}{m!\ p!} (a+n-1)_{m+p} \quad y_n (a+m+p,b;\ x)\ , \tag{5.3.18}$$

respectively.

5.4 GENERATING RELATIONS FOR $F_A^{(n)}$ AND ${}^{(k)}H_4^{(n)}$

We begin with the modified result of Exton [31] (cf. (5.3.12)). On replacing s and t by $s(1+z_1).......s(1+z_p)$ and $t(1+z_1).....(1+z_p)$ respectively in equation (5.3.12), multiplying both the sides by

$$\exp\ (-x_1 z_1 - ....-x_p z_p)$$

and using the result [94, p. 209 (9)]

$$\sum_{n=0}^{\infty} L_n^{(\alpha-n)}\ (x)\ t^n\ =\ (1+t)^{\alpha}\ e^{-xt}$$

we get

$$\exp\left[(s+t) \prod_{i=1}^{p} \{(1+z_i)\} - xt/s - x_1 z_1 - ....-x_p z_p\right]$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \sum_{k_1,\dots,k_p=0}^{\infty} \prod_{i=1}^{p} \{z_i^{k_i}\} \frac{s^m t^n}{(m+n)!} L^{(m)}(x)\, L^{(m+n-k_1)}(x_1)......L^{(m+n-k_p)}(x_p),\ |z|<1 \tag{5.4.1}$$

which is equivalent to

$$\exp\left[(s+t) \prod_{i=1}^{p} \{(1+z_i)\} - xt/s - x_1 z_1 - ... - x_p z_p\right]$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \sum_{k_1,\dots,k_p=0}^{\infty} \prod_{i=1}^{p} \left\{\frac{(1+m+n-k_i)_{k_i} z_i^{k_i}}{k_i!}\right\} \frac{s^m t^n}{m!\ n!}\ {}_1F_1\ [-n;\ m+1,\ x]$$

$$ {}_1F_1[-k_1;\ 1+m+n-k_1; x_1]..............{}_1F_1\,[-k_p;\ 1+m+n-k_p;\ x_p] \tag{5.4.2}$$

Now by replacing x, $x_1,\ldots, x_p$, s and t in (5.4.1) by xu, $x_1u,\ldots,x_pu$, su and tu respectively, multiplying both sides by u^{c-1} and taking Laplace transforms with help of the results [25, p.137(1)].

$$\int_0^\infty e^{-au} u^{c-1} du = \Gamma c \; a^{-c}, \; Re(a)>0, \; Re(c)>0. \tag{5.4.3}$$

and [94, p. 260(2(ii)]

$$\int_0^\infty e^{-pt} t^{a-1} L^{(\alpha_1)}(x_1t)\ldots\ldots L^{(\alpha_n)}(x_nt) = \Gamma a \; p^{-a} \binom{\alpha_1+m_1}{m_1} \ldots \binom{\alpha_n+m_n}{m_n}$$

$$F^{(n)}[a, -m_1,\ldots, -m_n; \alpha_1+1,\ldots, \alpha_n+1; x_1/p,\ldots, x_n/p], \quad Re(a)>0, \; Re(p)>0, \tag{5.4.4}$$

we get the following generating relation for Lauricella function $F_A^{(n)}$:

$$[1-(s+t)\prod_{i=1}^{p}(1+z_i) + xt/s + x_1z_1+\ldots\ldots\ldots+x_pz_p]^{-c}$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \sum_{k_1,\ldots,k_p=0}^{\infty} \prod_{i=1}^{p} \left\{ \binom{m+n}{k_i} z_i^{k_i} \right\} \frac{s^m \, t^n}{n!m!} (c)_{m+n}$$

$$F_A^{(p+1)}[m+n+c, -n, -k_1,\ldots, -k_p; m+1, 1+m+n-k_1,\ldots, 1+m+n-k_p; x, x_1,\ldots,x_p], \tag{5.4.5}$$

$Re(c)>0$, $|x|+|x_1|+\ldots+|x_p|<1$ and $|z|<1$.

For $s=t=((x/2+x_1z_1/2+\ldots+x_pz_p/2) / \prod_{i=1}^{p}(1+z_i))$, (5.4.5) reduces to

$$1 = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \sum_{k_1,\ldots,k_p=0}^{\infty} \prod_{i=1}^{p} \left\{ \binom{m+n}{k_i} z_i^{k_i} \right\} ((x+x_1z_1+\ldots+x_pz_p) / 2\prod_{i=1}^{p}(1+z_i))^{m+n} (c)_{m+n}$$

$$F_A^{(P+1)}[m+n+c, -n_1, -k_1,\ldots, -k_p; m+1, 1+m+n-k_1;\ldots,1+m+n-k_p; x, x_1,\ldots,x_p]. \tag{5.4.6}$$

Equation (5.4.5) establish an important formula whereby integral powers of $(x - x_1z_1 - \ldots - x_pz_p)$ may be expanded as multiple series of Lauricella function.

First we note

$$V(x, x_1, \ldots, x_p, s, t, c) = [\,1-(s+t)\prod_{i=1}^{p}(1+z_i)+xt/s+x_1z_1+\ldots+x_pz_p]^{-c}$$

gives

$$V(x, x_1, \ldots, x_p, (x/2+x_1z_1/2+\ldots+x_pz_p/2)/\prod_{i=1}^{p}(1+z_i)), (x/2+x_1z_1/2+\ldots+x_pz_p/2)/\prod_{i=1}^{p}(1+z_i))$$

and $\partial^r V/\partial t^r = (c)_r\,((x/s-\prod_{i=1}^{p}(1+z_i))^r[1-(s+t)\prod_{i=1}^{p}(1+z_i)+x_1z_1+\ldots+x_pz_p]^{-c-r}$

$$=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\sum_{k_1,\ldots,k_p=0}^{\infty}\prod_{i=1}^{p}\left\{\binom{m+n}{k_i} z_i^{k_i}\right\}(-n)_r\frac{s^m\,t^{n-r}}{m!\,n!}(c)_{m+n}$$

$$F_A^{(P+1)}[m+n+c, -n, -k_1, \ldots, -k_p;\ m+1,\ 1+m+n-k_1, \ldots, 1+m+n-k_p;\ x, x_1, \ldots, x_p] \quad (5.4.7)$$

When $s=t=((x/2+x_1z_1/2+\ldots+x_pz_p/2)/\prod_{i=1}^{p}(1+z_i))$, (5.4.7) yields

$$(x-x_1z_1-\ldots-x_pz_p)^r = (2^r(-n)_r/(c)_r)\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\sum_{k_1,\ldots,k_p=0}^{\infty}\prod_{i=1}^{p}\left\{\binom{m+n}{k_i} z_i^{k_i}\right\}(c)_{m+n}$$

$$\frac{(x+x_1z_1+\ldots+x_pz_p)^{m+n}}{\prod_{i=1}^{p}(1+z_i)^{m+n}\,2^{m+n}} F_A^{(P+1)}[m+n+c, -n, -k_1, \ldots, -k_p;\ m+1,\ 1+m+n-k_1, \ldots, 1+m+n-k_p;\ x, x_1, \ldots, x_p] \quad (5.4.8)$$

for $r=0, 1, 2\ldots$, $\mathrm{Re}(c)>0$.

A second set of expansions also exist which may be obtained in a similar manner by taking successive partial derivatives with respect to s of the generating relation (5.4.5) and letting $s=t=((x/2+x_1z_1/2+\ldots+x_pz_p/2)/\prod_{i=1}^{p}(1+z_i))$.

The general formula of these expansions has not, so far been obtained, and the expansions of powers of x up to x^2 are given below.

$$\frac{c(3x+x_1z_1+\ldots+x_pz_p)}{2}=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\sum_{k_1,\ldots,k_p=0}^{\infty}\prod_{i=1}^{p}\left\{\binom{m+n}{k_i}z_i^{k_i}\right\}m(c)_{m+n}$$

$$((x+x_1z_1+\ldots+x_pz_p)/2\prod_{i=1}^{p}(1+z_i))^{m+n}F_A^{(p+1)}[m+n+c,-n,-k_1,\ldots,-k_p;\ m+1,1+m+n-k_1,\ldots\ldots\ldots,$$

$$1+m+n-k_p;\ ;x,\ x_1,\ldots,x_p]\,, \tag{5.4.9}$$

$$\frac{c(c+1)(3x+x_1z_1+\ldots+x_pz_p)^2}{4}-2cx=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\sum_{k_1,\ldots,k_p=0}^{\infty}\prod_{i=1}^{p}\left\{\binom{m+n}{k_i}z_i^{k_i}\right\}m(m-1)\,(c)_{m+n}$$

$$((x+x_1z_1+\ldots+x_pz_p)/2\prod_{i=1}^{p}(1+z_i))^{m+n}F_A^{(p+1)}[m+n+c,-n,-k_1,\ldots,k_p;\ m+1,1+m+n-k_1,\ldots\ldots\ldots,$$

$$1+m+n-k_p;\ x,\ x_1,\ldots,x_p]\ . \tag{5.4.10}$$

We shall now generalize relation (5.4.5) and obtain generating function for Horn's function of (n+p+1) variables ${}^{(k)}H_4^{(n+p+1)}$. We recall [27, p.104 (3.5.4.5)]

$${}^{(k)}H_4^{(n)}[a,b_{k+1},\ldots,b_n;c_1,\ldots,c_n;\ x_1,\ldots,x_k,x_{k+1},\ldots,x_n]$$

$$=\frac{1}{\Gamma a}\int_0^{\infty}u^{a-1}e^{-pu}\prod_{i=1}^{k}{}_0F_1[-;c_i;\ x_iu^2]\prod_{i=k+1}^{n}{}_1F_1[b_i;\ c_i;\ x_iu]\,du\,. \tag{5.4.11}$$

On replacing x, $x_1,\ldots,x_p$, s and t by xu, $x_1u,\ldots,x_pu$, su and tu respectively in (5.4.2), multyplying both sides by

$$u^{c-1}\,e^{-pu}\prod_{i=1}^{k}{}_0F_1[-;\ c_i;\ y_iu^2]\prod_{i=k+1}^{n}{}_1F_1[b_i;\ c_i;\ y_iu]\,, \tag{5.4.12}$$

integrating the multiple series with respect to 'u' between the limits zero and infinity, using integral (5.4.11) and definition (1.12.1) and adjusting the parameters, we get

$$(\omega)^{-c}\,{}^{(k)}H_4^{(n)}[c, b_{k+1},\ldots,b_n; c_1,\ldots,c_n; y_1/\omega^2,\ldots, y_k/\omega^2, y_{k+1}/\omega,\ldots, y_n/\omega]$$

$$=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\sum_{k_1,\ldots,k_r=0}^{\infty}\prod_{i=1}^{p}\left\{\left(\frac{(1+m+n-k_i)_{k_i}}{k_i!}\right)z_i^{k_i}\right\}\frac{s^m t^n}{m!\,n!}(c)_{m+n}\,{}^{(k)}H_4^{(n+p+1)}[m+n+c,$$

$$b_{k+1},\ldots,b_n, -k_1,\ldots, -k_p, -n; c_1,\ldots,c_n, 1+m+n-k_1,\ldots,1+m+n-k_p, m+1; y_1,\ldots,y_n, x_1,\ldots,x_p, x] \quad (5.4.13)$$

where Re $(\omega)>0$, Re$(c)>0$ and

$$\omega = [\,1-(s+t)\prod_{i=1}^{p}(1+z_i) + xt/s + x_1z_1 + \ldots\ldots + x_pz_p\,] \quad (5.4.14)$$

Equation (5.4.5) is an interesting generalization of known results of Pathan and Yasmeen [68, p.241(2.1), p. 242(2.2),(2.3)]. For example if in (5.5.8), we put $x_i = k_i = z_i = 0$, i=1,2,...p, and use the results [27,p.215(5.9.2) and p.216(6.9.6)] then we arrive at the result [68, p. 241(2.1)].

$$(1-s-t+xt/s)^{-c} = \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{(c)_{2n+m}\, s^m t^n (1-x)^n}{(m+n)!\; n!}\,{}_2F_1[-n, 1-c-n; 1-m-2n-c; 1/(1-x)] \quad (5.4.15)$$

Obviously the results [68, p.242 (2.2) and (2.3)] follows from (5.5.17). The results (5.4.9) and (5.4.10) include recent results due to Pathan and Yasmeen [68,p.242(2.6) and (2.7)] as special cases for $x_i=k_i=z_i=0$, i=1,2,...p.

Further, if we put $x_i=k_i=z_i=0$, i=1,2,...p, in (5.4.13), it reduces to a known result of Pathan and Kamarujjama [65, p.33(2.4)].

On setting k=0 and using the relationship [65,p.32(1.7)]

$${}^{(0)}H^{(n)}[a,b_1,\ldots,b_n; c_1,\ldots, c_n; x_1,\ldots,x_n] = F_A^{(n)}[a,b_1,\ldots,b_n; c_1,\ldots,c_n; x_1,\ldots,x_n] \quad (5.4.16)$$

equation (5.4.13) reduces to

$(\omega)^{-c} F^{(n)}[c, b_1,\ldots,b_n; c_1,\ldots,c_n; y_1/\omega,\ldots, y_n/\omega]$

$$=\sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \sum_{k_1,\ldots,k_p=0}^{\infty} \prod_{i=1}^{p}\left\{\frac{(1+m+n-k_i)_{k_i}}{k_i} z_i^{k_i}\right\}\frac{s^m t^n}{m!n!} F_A^{(n+p+1)}[m+n+c, b_1,\ldots, b_n, -k_1,\ldots,$$

$$-k_p, -n; c_1,\ldots,c_n, 1+m+n-k_1,\ldots,1+m+n-k_p, m+1; y_1,\ldots, y_n, x_1,\ldots, x_p, x] \qquad (5.4.17)$$

where ω is given by (5.4.14). Note that for $x_i=k_i=z_i=0$, $i=1,2,\ldots p$, (5.4.17) reduces to another result due to Pathan and Yasmeen [68,p.143(3.3)].

$(\omega')^{-c} F^{(n)}[c, b_1,\ldots,b_n; c_1,\ldots,c_n; y_1/\omega',\ldots, y_n/\omega']$

$$=\sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{s^m t^n}{m!n!} F_A^{(n+1)}[m+n+c,b_1,\ldots, b_n, -n;c_1,\ldots,c_n, m+1; y_1,\ldots, y_n, x_1,\ldots, x_p, x] \qquad (5.4.18)$$

where $\omega' = (1\text{-}s\text{-}t+xt/s)$.

CHAPTER-VI

ON REPRESENTATIONS OF THE GENERALIZED VOIGT FUNCTIONS

6.1 INTRODUCTION

The familiar Voigt functions [74]

$$K(x,\ y) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \exp\ (-yt-\tfrac{1}{4}t^2)\ \cos(xt)\ dt \tag{6.1.1}$$

$$L(x,\ y) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \exp\ (-yt-\tfrac{1}{4}t^2)\ \sin(xt)\ dt \tag{6.1.2}$$

$$(-\infty < x < \infty\ ;\ y > 0),$$

occur frequently in a wide variety of physical problems such as astrophysical spectroscopy and the theory of neutron reactions. Furthermore the function

$$K(x,y) + iL(x,y) \tag{6.1.3}$$

is, except for a numerical factor, identical to the so-called plasma dispersion function which is tabulated by Fried and Conte [34] and Fettis et al. [33]. In many given physical problems, a numerical or analytical evaluation of the Voigt functions is required. For an excellent review of various mathematical properties and computational methods concerning the Voigt functions see, for example, Armstrong and Nicholls [6] and Haubold and John [42].

On other hand, it is well known that, Bessel functions are closely associated with problems possessing circular or cylindrical symmetry. For example, they arise in the theory of electromagnetism and in the study of free vibrations of a circular membrance [55].

Motivated by the relationships.

$$J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}} \cos(z) \text{ and } J_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sin(z) \tag{6.1.4}$$

where

$$J_v(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (\tfrac{1}{2}z)^{v+2m}}{m!\ \Gamma(v+m+1)}, \quad |z|<\infty, \tag{6.1.5}$$

Srivastava and Miller [95] established a link of Bessel functions with the generalized Voigt function in the form

$$V_{\mu,v}(x,y) = \sqrt{\frac{x}{2}} \int_0^{\infty} t^{\mu} \exp(-yt-\tfrac{1}{4}t^2)\ J_v(xt)\ dt \tag{6.1.6}$$

$$(x,\ y \in R^+;\ R(\mu+v) >-1),$$

so that

$$V_{\frac{1}{2},-\frac{1}{2}}(x,y) = K(x,y) \text{ and } V_{\frac{1}{2},\frac{1}{2}}(x,y) = L(x,y). \tag{6.1.7}$$

Subsequently, following the work of Srivastava and Miller [95] closely, Klusch [52] proposed an integral representation of the Voigt functions in the form

$$\Omega_{\mu,v}[x,y,z] = \sqrt{\frac{x}{2}} \int_0^{\infty} t^{\mu} \exp(-yt-zt^2)\ J_v(xt)\ dt \tag{6.1.8}$$

$$(x,\ y,\ z \in R^+ ;\ Re\ (\mu+v)>-1\ ).$$

By comparing (6.1.6) and (6.1.8), we find that

$$\Omega_{\mu,v}[x,y,z] = (2\sqrt{z})^{-\mu-\frac{1}{2}}\ V_{\mu,v}\left(\frac{x}{2\sqrt{z}},\ \frac{y}{2\sqrt{z}}\right). \tag{6.1.9}$$

The relations (6.1.6) and (6.1.8) are, in fact, unification (and generalization) of the Voigt functions K(x,y) and L(x,y).

In an attempt to generalize the work of Srivastava and Miller [59], Siddiqui (cf. [80] and [89]) studied the following unification (and generalization) of the Voigt functions K(x,y) and L(x,y) in the form

$$\Omega^{\mu}_{\eta,v,\lambda}\ [x,y] = \sqrt{\frac{x}{2}} \int_0^{\infty} t^{\eta} \exp(-yt-\tfrac{1}{4}t^2)\ J^{\mu}_{v,\lambda}(xt)\ dt \qquad (6.1.10)$$

$$(\ x,\ y,\ \mu \in R^{+}\ ;\quad Re(\eta+v+2\lambda)>-1\ ),$$

where

$$J^{\mu}_{v,\lambda}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m\ (\tfrac{1}{2}z)^{v+2\lambda+2m}}{\Gamma(\lambda+m+1)\ \Gamma(v+\lambda+m\mu+1)}\ . \qquad (6.1.11)$$

Srivastava, Pathan and Kamarujjama [96] have studied and investigated a slightly modified form of formula (6.1.10) in the form given below

$$\Omega^{\mu}_{\eta,v,\lambda}\ [x,y,z] = \sqrt{\frac{x}{2}} \int_0^{\infty} t^{\eta} \exp(-yt-zt^2)\ J^{\mu}_{v,\lambda}(xt)\ dt \qquad (6.1.12)$$

$$(\ x,\ y,\ z,\ \mu \in R^{+}\ ;\quad Re(\eta+v+2\lambda)>-1\ ).$$

For the purpose of our present study, we recall the definition of the generalized Bessel function $J_n(z,y;s)$ in the form [18,p.3(1.1)]

$$J_n(z,y;s) = \sum_{m=-\infty}^{\infty} s^m\ J_{n-2m}(z)\ J_m(y)\ ,\ (\ z\ ,\ y\ )\in R \qquad (6.1.13)$$

Function $J_n(z,y;s)$ has the following generating function [18,p.11(2.4.2)]

$$\exp\left[\frac{z}{2}\left(t-\frac{1}{t}\right)+\frac{y}{2}\left(st^2-\frac{1}{st^2}\right)\right] = \sum_{n=-\infty}^{\infty} t^n\ J_n(z,y;s) \qquad (6.1.14)$$

$$(\ s\ ,\ t\ ) \neq 0.$$

As far as the applications of $J_n(x,y;s)$ are concerned, they frequently arise in physical problems of quantum electrodynamics and optics, the emission of electromagnetic radiation, scattering of laser radiation from free or weakly bounded electrons, the generation of betatron harmonics and the mutual absorption of two nonparallel classical photon fields through the production of electron pairs (cf. [18],[19] and [20]). This chapter aims at some representation and unification of the Voigt functions K(x,y) and L(x,y) which play a rather important role in several diverse fields of physics. We derive in the present chapter several representations of these functions in terms of series and integrals which are specially useful in situations when the parameters and variables take on particular values. In section 6.2 of this chapter we will focus our attention on finding different expansions forms of an integral formula involving the product of three Bessel polynomials. In section 6.3 we will explore the possibility of considering bilinear representations of the generalized Voigt function $\Omega^{\mu}_{\eta,\nu,\lambda}[x,y,z]$ with the help of the generalized Bessel function $J_n(x,y;s)$.

In the remaining parts of this chapter we will establish a new partly bilateral and partly unilateral representation of the generalized Voigt functions $\Omega^{\mu}_{\eta,\nu,\lambda}[x,y,z]$ and some interesting generating relations by means of certain explicit representation of the generalized Voigt function $\Omega_{\mu,\nu}[x,y,z]$.

6.2. INTEGRALS

The aim of this section is to obtain different expansions involving the confluent series $\Psi_2^{(n)}$ [94,p.62], Kampe' de Fe'riet's function $F^{A:B;B'}_{C:D;D'}$ of two variables [94,p.65], and Srivastava's function $F^{(3)}$ [94,p.69].

Indeed, we first establish an integral involving the product of three Bessel polynomials in the form

$$I^{\sigma,\delta,v,\mu}_{(z,y,x)} = \int_0^\infty u^\eta \exp(-qu-\gamma u^2)\ J_\sigma(zu)\ J_\delta(yu)\ J^\mu_{v,\lambda}(xu)\ du \qquad (6.2.1)$$

$$= z^\sigma\ y^\delta\ x^v\ K \sum_{l=0}^{\infty} \frac{(-x^2/4\gamma)^l}{\Gamma(\lambda+l+1)\ \Gamma(v+\lambda+\mu l+1)}$$

$$\Big\{\Gamma(\tfrac{1}{2}(\eta+2l+2\lambda+v+\sigma+\delta+1))\ \Psi_2^{(3)}\Big[\tfrac{1}{2}(\eta+2l+2\lambda+v+\sigma+\delta+1);\ n+1,m+1,\tfrac{1}{2};\ \frac{-z^2}{4\gamma},\frac{-y^2}{4\gamma},\frac{q^2}{4\gamma}\Big]$$

$$-\frac{q}{\sqrt{\gamma}}\ \Gamma(\tfrac{1}{2}(\eta+2l+2\lambda+v+\sigma+s+2))\Psi_2^{(3)}\Big[\tfrac{1}{2}(\eta+2l+2\lambda+v+\sigma+\delta+2);\ n+1,m+1,3/2;\ \frac{-z^2}{4\gamma},\frac{-y^2}{4\gamma},\frac{q^2}{4\gamma}\Big]\Big\} \qquad (6.2.2)$$

$$=\Big(1-\frac{y^2}{z^2}\Big)^{\sigma+\delta+1} z^\sigma y^\delta x^v\ K \sum_{l=0}^{\infty} \frac{(-x^2/4\gamma)^l}{\Gamma(\lambda+l+1)\ \Gamma(v+\lambda+\mu l+1)}\Big\{\Gamma(\tfrac{1}{2}(\eta+2l+2\lambda+v+\sigma+\delta+1))$$

$$F^{(3)}\left[\begin{matrix} -::\ \sigma+\delta+1,\ \delta+1;\ -;\tfrac{1}{2}(\eta+2l+2\lambda+v+\sigma+\delta+1):\text{———};-;-; \\ -::\text{———};\ -;\text{———}:\sigma+\delta+1,\sigma+1,\delta+1;\delta+1;\tfrac{1}{2}; \end{matrix}\ \frac{-z^2}{4\gamma}\Big(1-\frac{y^2}{z^2}\Big),\ \frac{y^2}{z^2},\ \frac{q^2}{4\gamma}\right]$$

$$-\frac{q}{\sqrt{\gamma}}\ \Gamma(\tfrac{1}{2}(\eta+2l+2\lambda+v+\sigma+\delta+2))$$

$$F^{(3)}\left[\begin{matrix} -::\ \sigma+\delta+1,\ \delta+1;\ -;\ \tfrac{1}{2}(\eta+2l+2\lambda+v+\sigma+\delta+2):\text{———};-;-; \\ -::\text{———};\ -;\text{———}:\sigma+\delta+1,\sigma+1,\delta+1;\delta+1;3/2; \end{matrix}\ \frac{-z^2}{4\gamma}\Big(1-\frac{y^2}{z^2}\Big),\ \frac{y^2}{z^2},\ \frac{q^2}{4\gamma}\right]\Big\} \qquad (6.2.3)$$

$$I^{\sigma,s,v,\mu}_{(y,y,x)} = y^{\sigma+\delta} x^{v} K \sum_{l=0}^{\infty} \frac{(-x^2/4\gamma)^l}{\Gamma(\lambda+l+1)\,\Gamma(v+\lambda+\mu l+1)} \Big\{ \Gamma\big(\tfrac{1}{2}(\eta+2l+2\lambda+v+\sigma+\delta+1)\big)$$

$$F^{1:2;0}_{0:3;1}\left[\begin{matrix} \tfrac{1}{2}(\eta+2l+2\lambda+v+\sigma+\delta+1 : \tfrac{1}{2}(\sigma+\delta+1),\ \tfrac{1}{2}(\sigma+\delta+2);-; \\ \rule{4cm}{0.4pt} : \sigma+1,\ \delta+1,\ \sigma+\delta+1 \quad ;\tfrac{1}{2}; \end{matrix}\ -\frac{y^2}{\gamma}, \frac{q^2}{4\gamma}\right]$$

$$-\ \frac{q}{\sqrt{\gamma}}\ \Gamma\big(\tfrac{1}{2}(\eta+2l+2\lambda+v+\sigma+\delta+2)\big)$$

$$F^{1:2;0}_{0:3;1}\left[\begin{matrix} \tfrac{1}{2}(\eta+2l+2\lambda+v+\sigma+\delta+2):\tfrac{1}{2}(\sigma+\delta+1),\ \tfrac{1}{2}(\sigma+\delta+2);-; \\ \rule{4cm}{0.4pt} :\sigma+1,\ \delta+1,\ \sigma+\delta+1\ ;3/2; \end{matrix}\ -\frac{y^2}{\gamma}, \frac{q^2}{4\gamma}\right] \Big\} \qquad (6.2.4)$$

where $$K = \frac{\gamma^{-\frac{1}{2}(\eta+2\lambda+\sigma+\delta+v+1)}}{\Gamma(\sigma+1)\ \Gamma(\delta+1)\ \Gamma(v+1)\ 2^{2\lambda+\sigma+\delta+v+1}} .$$

<u>**Derivations of (6.2.2) to (6.2.4):**</u>

To establish (6.2.2), expressing $J_\sigma(zu)$, $J_\delta(yu)$ and $J^{\mu}_{v,\lambda}(xu)$ in series, expanding exp(-qu) in the series form

$$\sum_{k=0}^{\infty} \frac{(-qu)^k}{k!} , \qquad (6.2.5)$$

integrating term by term with the help of the result (cf. [96,p.58(3.4)])

$$\int_0^{\infty} x^{s-1} \exp(-\alpha x^2)dx = \tfrac{1}{2}\ \alpha^{-\frac{1}{2}s}\ \Gamma(\tfrac{1}{2}s) , \qquad (6.2.6)$$

$$(\operatorname{Re}(s)>0 \ , \ \operatorname{Re}(x)>0) ,$$

and then separating the k-series into its even and odd terms [94,p.200(3)], we arrive at (6.2.2). If we use the relation [23,p.11(47) and p.64(23)]

$$(\alpha+1)\ (\delta+1)\ J_\sigma(zu)\ J_\delta(yu)=\left(\frac{zu}{2}\right)^\sigma\left(\frac{yu}{2}\right)^\delta\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{zu}{2}\right)^{2m}}{m!\ (\sigma+1)_m}\left(1-\frac{y^2}{z^2}\right)^{\sigma+\delta+2m+1}$$

$$_2F_1\ [\delta+m+1,\ \sigma+\delta+m+1;\ \delta+1;\ y^2/z^2]. \quad (6.2.7)$$

replace $J^{\mu}_{v,\lambda}(xu)$ by its series representation (6.1.11), expand exp(-qu) as in (6.2.5), integrate term by term with the help of the result (6.2.6) and then separate the k-series into its even and odd terms, we get (6.2.3). On other hand, if in (6.2.1), we set z=y, apply the relation [5,p.11(49)]

$$J_\sigma(yu)\ J_\delta(yu)=\frac{(xu/s)^{\sigma+\delta}}{\Gamma\alpha+1\ \Gamma\delta+1}\ {}_2F_3\ [\tfrac{1}{2}(\sigma+\delta+1),\tfrac{1}{2}(\sigma+\delta+2);\sigma+1,\delta+1,\sigma+\delta+1;\ -y^2u^2]\ , \quad (6.2.8)$$

express $J^{\mu}_{v,\lambda}(xu)$ in its series form (6.1.11), expand exp(-qu) as in (6.2.5), integrate term by term with the help of the result (6.2.6) and then separate the k-series into its even and odd terms, we arrive at (6.2.4).

6.3. BILINEAR REPRESENTATIONS FOR THE GENERALIZED VOIGT FUNCTIONS $\Omega^{\mu}_{\eta,v,\lambda}[x,y,z]$

As an application of the results obtained in section 6.2, the definition of the generalized Bessel function $J_n(z,y;s)$ (cf.(5.1.13)) and its generating relation (6.1.14), we will explore in this section the possibility of considering bilinear representations of the generalized Voigt functions $\Omega^{\mu}_{\eta,v,\lambda}[x,y,z]$ in term of more familiar special functions.

On replacing the generalzied Bessel function $J_n(z,y;s)$ in equation (6.1.14) by the series representation (6.1.13), we get

$$\exp\left[\frac{z}{2}(t-\frac{1}{t})+\frac{y}{2}(st^2-\frac{1}{st^2})\right] = \sum_{n,m=-\infty}^{\infty} t^n s^m J_{n-2m}(z) J_m(y) . \quad (6.3.1)$$

Now, starting from (6.3.1), multiplying both sides by

$$\sqrt{\frac{x}{2}} \int_0^{\infty} u^{\eta} \exp(-qu-\gamma u^2) J^{\mu}_{v,\lambda}(xu) , (\gamma>0), \quad (6.3.2)$$

replacing z and y by zu and yu respectively, integrating the resulting expression with respect to u over the semi-infinite interval $(0,\infty)$, and using the integral representation (6.1.10), we obtain

$$\Omega^{\mu}_{\eta,v,\lambda}\left[x,q - \frac{z}{2}(t-\frac{1}{t}) - \frac{y}{2}(st^2-\frac{1}{st^2}), \gamma\right]$$

$$= \sum_{n,m=-\infty.}^{\infty} t^n s^m \int_0^{\infty} u^{\eta} \exp(-qu-\gamma u^2) J_{n-2m}(zu) J_m(yu) J^{\mu}_{v,\lambda}(xu) du, \quad (6.3.3)$$

$$\left(x,q,\gamma,\mu\in R^+;\ R(\eta+v+2\lambda+n-m)>-1;\ \mathrm{Re}\left[q-\frac{z}{2}(t-\frac{1}{t})-\frac{y}{2}(st^2-\frac{1}{st^2})\right]>0\right).$$

Since the generalized Voigt functions $\Omega^{\mu}_{\eta,v,\lambda}[x,y,z]$ can be expressed in terms of integral representation involving the Bessel function $J_n(x,y;u)$, the properties of this last function assume noticeable importance. Indeed, each of these properties will naturally lead to various other needed properties of the generalized Voigt functions $\Omega^{\mu}_{\eta,v,\lambda}[x,y,z]$.

By means of integral formulae (6.2.2), (6.2.3) and (6.2.4) one can obtain the following bilinear representations:

$$\Omega^{\mu}_{\eta,\nu,\lambda}\left[x,\ q-\frac{z}{2}\left(t-\frac{1}{t}\right)-\frac{y}{2}\left(st^{2}-\frac{1}{st^{2}}\right),\ \gamma\right]$$

$$= W\sum_{n,m=-\infty}^{\infty}\frac{(zt/2\sqrt{\gamma})^{n}\ (yt^{2}s/2\sqrt{\gamma})^{m}}{\Gamma(n+1)\ \Gamma(m+1)}\ \sum_{l=0}^{\infty}\frac{(-x^{2}/4\gamma)^{l}}{\Gamma(\lambda+l+1)\Gamma(\nu+\lambda+\mu l+1)}$$

$$\Big\{\Gamma(\tfrac12(\eta+\nu+2\lambda+n+m+2l+1))\ \Psi_{2}^{(3)}\left[\tfrac12(\eta+\nu+2\lambda+n+m+2l+1);n+1,m+1,\tfrac12;\frac{-z^{2}}{4\gamma},\frac{-y^{2}}{4\gamma},\frac{q^{2}}{4\gamma}\right]$$

$$-\frac{q}{\sqrt{\gamma}}\Gamma(\tfrac12(\eta+\nu+2\lambda+n+m+2l+2))\ \Psi_{2}^{(3)}\left[\tfrac12(\eta+\nu+2\lambda+n+m+2l+2);\ n+1,m+1,\tfrac32;\frac{-z^{2}}{4\gamma},\frac{-y^{2}}{4\gamma},\frac{q^{2}}{4\gamma}\right]\Big\}$$

(6.3.4)

$$\Omega^{\mu}_{\eta,\nu,\lambda}\left[x,\ q-\frac{z}{2}\left(t-\frac{1}{t}\right)-\frac{y}{2}\left(st^{2}-\frac{1}{st^{2}}\right),\ \gamma\right]$$

$$= W\left(1-\frac{y^{2}}{z^{2}}\right)\sum_{n,m=-\infty}^{\infty}\frac{\left[\frac{z}{2\sqrt{\gamma}}\left(1-\frac{y^{2}}{z^{2}}\right)\right]^{n}\left[\frac{yt^{2}s}{2\sqrt{\gamma}}\left(1-\frac{y^{2}}{z^{2}}\right)\right]^{m}}{\Gamma(n+1)\quad\Gamma(m+1)}\sum_{l=0}^{\infty}\frac{(-x^{2}/4\gamma)^{l}}{\Gamma(\lambda+l+1)\Gamma(\nu+\lambda+\mu l+1)}$$

$$\Big\{\Gamma(\tfrac12(\eta+\nu+2\lambda+n+m+2l+1))\ F^{(3)}\left[\begin{array}{l}-::n+m+1,m+1;-;\ \tfrac12(\eta+\nu+2\lambda+n+m+2l+1):\text{———};\text{———};-;\\ -::\text{———};-;\text{———}::n+m+1,m+1,n+1;m+1;\tfrac12;\end{array}\ \frac{-z^{2}}{4\gamma}\left(1-\frac{y^{2}}{z^{2}}\right),\ \frac{y^{2}}{z^{2}},\ \frac{q^{2}}{4\gamma}\right]$$

$$-\frac{q}{\sqrt{\gamma}}\Gamma(\tfrac12(\eta+\nu+2\lambda+n+m+2l+2))\ F^{(3)}\left[\begin{array}{l}-::n+m+1,\ m+1;-;\tfrac12(\eta+\nu+2\lambda n+m+2l+2):\text{———};\text{—};\text{—};\\ -::\text{———};-;\text{———}::n+m+1,\ m+1,\ n+1;m+1;3/2;\end{array}\ \frac{-z^{2}}{4\gamma}\left(1-\frac{y^{2}}{z^{2}}\right),\ \frac{y^{2}}{z^{2}},\ \frac{q^{2}}{4\gamma}\right]\Big\}$$

(6.3.5)

$$\Omega^{\mu}_{\eta,\nu,\lambda}\left[x,\ q-\frac{y}{2}\left(t-\frac{1}{t}\right)-\frac{y}{2}\left(st^2-\frac{1}{st^2}\right),\ \gamma\right]=W\sum_{n,m=-\infty}^{\infty}\frac{(yt/2\sqrt{\gamma})^n\,(yt^2s/2\sqrt{\gamma})^m}{\Gamma(n+1)\ \Gamma(m+1)}$$

$$\sum_{l=0}^{\infty}\frac{(-x^2/4\gamma)^l}{\Gamma(\lambda+l+1)\ \Gamma(\nu+\lambda+\mu l+1)}\ \Big\{\ \Gamma(\tfrac{1}{2}(\eta+\nu+2\lambda+n+m+2l+1))$$

$$F^{1:2;0}_{0:3;1}\left[\begin{matrix}\tfrac{1}{2}(\eta+\nu+1+n+m+2l+1):\tfrac{1}{2}(n+m+1),\tfrac{1}{2}(n+m+2);-; & \dfrac{-y^2}{\gamma},\ \dfrac{q^2}{4\gamma}\\ \text{———}:n+1,m+1,n+m+1\quad ;\tfrac{1}{2}; & \end{matrix}\right]$$

$$-\frac{q}{\sqrt{\gamma}}\ \Gamma(\tfrac{1}{2}(\eta+\nu+2\lambda+n+m+2l+2))$$

$$F^{1:2;0}_{0:3;1}\left[\begin{matrix}\tfrac{1}{2}(\eta+\nu+2\lambda+n+m+2l+2):\ \tfrac{1}{2}(n+m+1),\tfrac{1}{2}(n+m+2);-; & \dfrac{-y^2}{\gamma},\ \dfrac{q^2}{4\gamma}\\ \text{———}:\ n+1,m+1,n+m+1\quad ;3/2; & \end{matrix}\right]\Big\},\quad(6.3.6)$$

where $\quad W=\dfrac{x^{\nu+2\lambda+\frac{1}{2}}\ \gamma^{-\frac{1}{2}(\eta+\nu+2\lambda+1)}}{2^{\nu+2\lambda+3/2}}$.

In addition, setting $\lambda=\mu-1=0$ in equations (6.3.4), (6.3.5) and (6.3.6), one gets the following bilinear representations of the generalized Voigt function $\Omega_{\eta,\nu}[x,y,z]$ in the form

$$\Omega_{\eta,\nu}\left[x,q-\frac{z}{2}\left(t-\frac{1}{t}\right)-\frac{y}{2}\left(st^2-\frac{1}{st^2}\right),\gamma\right]=R\sum_{n,m=-\infty}^{\infty}\frac{(zt/2\sqrt{\gamma})^n\ (yt^2s/2\sqrt{\gamma})^m}{\Gamma(n+1)\ \Gamma(m+1)}$$

$$\Big\{\Gamma(\tfrac{1}{2}(\eta+\nu+m+n+1));\Psi_2^{(4)}\left[\tfrac{1}{2}(\eta+\nu+m+n+1);n+1,m+1,\nu+1,\tfrac{1}{2};\frac{-z^2}{4\gamma},\ \frac{-y^2}{4\gamma},\frac{-x^2}{4\gamma},\frac{q^2}{4\gamma}\right]$$

$$-\frac{q}{\sqrt{\gamma}}\ \Gamma(\tfrac{1}{2}(\eta+\nu+n+m+2))\ \Psi_2^{(4)}\left[\tfrac{1}{2}(\eta+\nu+m+2);n+1;m+1,\nu+1,\frac{3}{2};\ \frac{-z^2}{4\gamma},\frac{-y^2}{4\gamma},\frac{-x^2}{4\gamma},\frac{q^2}{4\gamma}\right]\Big\},(6.3.7)$$

$$\Omega_{\eta,v}[x,q-\frac{z}{2}(t-\frac{1}{t})-\frac{y}{2}(st^2-\frac{1}{st^2}),\gamma]$$

$$= R\left(1-\frac{y^2}{z^2}\right)\sum_{n,m=-\infty}^{\infty}\frac{\left[\frac{zt}{2\sqrt{\gamma}}\left(1-\frac{y^2}{z^2}\right)\right]^n\left[\frac{yt^2s}{2\sqrt{\gamma}}\left(1-\frac{y^2}{z^2}\right)\right]^m}{\lfloor n+1\ \lfloor m+1}\Big\{\Gamma(\tfrac{1}{2}(\eta+v+n+m+1))$$

$$F_p^{(4)}\left[\begin{matrix}-::-;-;\ \tfrac{1}{2}(\eta+v+m+n+1);-:\ n+m+1,m+1;-;-;-:\text{———};\text{—};\text{—};-;\\ -::-;-;\text{———};-:\text{———};-;-;-:n+m+1,m+1,n+1;m+1;v+1;\tfrac{1}{2};\end{matrix}\right.$$

$$\left.\frac{z^2}{4\gamma}(1-\frac{y^2}{z^2})\ \frac{y^2}{z^2}\quad \frac{-x^2}{4\gamma}\ \frac{q^2}{4\gamma}\right]$$

$$-\frac{q}{\sqrt{\gamma}}\,\Gamma(\tfrac{1}{2}(\eta+v+n+m+2))$$

$$F_p^{(4)}\left[\begin{matrix}-::-;-;\ \tfrac{1}{2}(\eta+v+m+n+1);-:\ n+m+1,m+1;-;-;-:\text{———};\text{—};\text{—};\text{—};\\ -::-;-;\text{———};-:\text{———};-;-;-:n+m+1,m+1,n+1;m+1;v+1;{}^3/_2;\end{matrix}\right.$$

$$\left.\frac{z^2}{4\gamma}(1-\frac{y^2}{z^2}),\ \frac{y^2}{z^2},\quad \frac{-x^2}{4\gamma},\ \frac{q^2}{4\gamma}\right]\Big\}, \qquad (6.3.8)$$

where $F_p^{(4)}$ is Pathan's function defined by (1.9.1), and

$$\Omega_{\eta,v}[x,q-\frac{y}{2}(t-\frac{1}{t})-\frac{y}{2}(st^2-\frac{1}{st^2}),\gamma]$$

$$= R\sum_{n,m=-\infty}^{\infty}\frac{(zt/2\sqrt{\gamma})^n\ (yt^2s/2\sqrt{\gamma})^m}{\lfloor n+1\ \lfloor m+1}\Big\{\Gamma(\tfrac{1}{2}(\eta+v+m+n+1))$$

$$F^{(3)}\left[\begin{matrix}\tfrac{1}{2}(\eta+v+m+n+1)::-;-;-:\ \tfrac{1}{2}(n+m+1),\ \tfrac{1}{2}(n+m+2);\text{—};-;\\ \text{———}::-;-;-:\ n+1,\ m+1,\ n+m+1\quad ;v+1;\tfrac{1}{2};\end{matrix}\ \frac{-y^2}{\gamma},\ \frac{-x^2}{4\gamma},\ \frac{q^2}{4\gamma}\right]$$

$$-\frac{q}{\sqrt{\gamma}}\,\Gamma(\tfrac{1}{2}(\eta+v+m+n+2))$$

$$F^{(3)}\left[\begin{array}{l} \tfrac{1}{2}(\eta+v+m+n+2)::-;-;-:\ \tfrac{1}{2}(n+m+1),\ \tfrac{1}{2}(n+m+2);—;-;\quad \dfrac{-y^2}{\gamma},\ \dfrac{-x^2}{4\gamma},\ \dfrac{q^2}{4\gamma} \\ \rule{3cm}{0.4pt}::-;-;-:\ n+1,\ m+1,\ n+m+1\ ;\ v+1;3/2; \end{array}\right]\Bigg\} \quad (6.3.9)$$

where $R=\left(x^{v+\frac{1}{2}}\ \gamma^{\frac{1}{2}(\eta+v+1)}/(2^{v+3/2}\,\Gamma(v+1)\right)$.

When s=t=1, it is not difficult to observe that

$$\Omega_{\eta,v}\left[x,q-\frac{z}{2}\left(t-\frac{1}{t}\right)-\frac{y}{2}\left(st^2-\frac{1}{st^2}\right),\gamma\right] = \Omega_{\eta,v}[x,q,\gamma]\ . \qquad (6.3.10)$$

Moreover, putting s=t=1, $\gamma=\frac{1}{4}$ and $\eta = v = \pm\frac{1}{2}$, equation (6.3.7) reduces to

$$K(x,q) = \frac{1}{\sqrt{\pi}} \sum_{n,m=-\infty}^{\infty} \frac{z^n\, y^m}{\Gamma(n+1)\Gamma(m+1)} \Big\{ \Gamma(\tfrac{1}{2}(n+m+1))$$

$$\Psi_2^{(4)}\left[\tfrac{1}{2}(n+m+1);n+1,m+1,\tfrac{1}{2},\tfrac{1}{2};-z^2,-y^2,-x^2,q^2\right]$$

$$-2q\,\Gamma(\tfrac{1}{2}(n+m+2))\ \Psi_2^{(4)}\left[\tfrac{1}{2}(n+m+2);n+1,m+1,3/2,\tfrac{1}{2};-z^2,-y^2,-x^2,q^2\right]\Big\}, \quad (6.3.11)$$

$$L(x,q) = \frac{2x}{\sqrt{\pi}} \sum_{n,m=-\infty}^{\infty} \frac{z^n\, y^m}{\Gamma(n+1)\Gamma(m+1)} \Big\{ \Gamma(\tfrac{1}{2}(n+m+2))$$

$$\Psi_2^{(4)}\left[\tfrac{1}{2}(n+m+2);n+1,m+1,3/2,\tfrac{1}{2};-z^2,-y^2,-x^2,q^2\right]$$

$$-2q\,\Gamma(\tfrac{1}{2}(n+m+3))\ \Psi_2^{(4)}\left[\tfrac{1}{2}(n+m+3);n+1,m+1,3/2,\tfrac{1}{2};-z^2,-y^2,-x^2,q^2\right]\Big\}. \quad (6.3.12)$$

Similarly other representations of K(x,y) and L(x,y) can be obtained from equations (6.3.8) and (6.3.9). For $z,y\to 0$, equations {(6.3.4),(6.3.5) and (6.3.6)} and {(6.3.7), (6.3.8) and (6.3.9)} reduce to known results due to Srivastava et al. [96] and Klusch [52] respectively. It is also not difficult to verify that known results due to Srivastava and Miller [95, Equations (10) and (11)] and Exton [29, Equation (8) and (9)] are special cases of our results of this section.

6.4 PARTLY BILATERAL AND PARTLY UNILATERAL REPRESENTATIONS FOR THE GENERALIZED VOIGT FUNCTIONS $\Omega^{\mu}_{\eta,\nu,\lambda}[x,y,z]$

We begin by recalling the modified result of Exton due to Pathan and Yasmeen ([68] and [69]) :

$$\exp(s+t-zt/s) = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{s^m t^n}{m!\, n!} {}_1F_1\,[-n;\ m+1;\ z], \tag{6.4.1}$$

where $m^*=\max(0,-m)$.From equation (5.4.2), we deduce the following formula :

$$\exp\Big((s+t)(1+\omega)-zt/s-y\omega\Big) = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \sum_{p=0}^{\infty} \frac{s^m t^n \omega^p}{m!\, n!\, p!} (1+m+n-p)_p$$

$${}_1F_1\,[-n;m+1;z]\ {}_1F_1\,[-p;1+m+n-p;y]\quad ,\quad |\omega|<1. \tag{6.4.2}$$

On replacing s, t, z and y by su^2, tu^2, zu^2 and yu^2 respectively, multiplying both sides of (6.4.2) by

$$u^{\eta} \exp(-qu-\gamma u^2)\ J^{\mu}_{\nu,\lambda}(xu)\quad ,\ (\gamma)>0, \tag{6.4.3}$$

integrating with respect to u over the semi-infinite interval $(0,\infty)$, using the integral representation (6.1.10) and the series representation (6.1.11) and expanding exp(-qu) as in (6.2.5), we can integrate the resulting expression term by term by means of Millen transform (cf. [96,p.58 (3.4)].)

$$\int_0^{\infty} u^{s-1} \exp(-\alpha u^2)\ {}_1F_1\,[a;b;xu^2]\ {}_1F_1\,[c;d;yu^2]$$

$$= \tfrac{1}{2}\alpha^{-\frac{1}{2}(s)}\, \Gamma\!\left(\tfrac{1}{2}s\right) F_2\,[\tfrac{1}{2}s;\ a,\ c;\ b,\ d;\ \frac{x}{\alpha},\ \frac{y}{\alpha}\ ], \tag{6.4.4}$$

$$(\ R(s)>0,\ Re(\alpha)>0 \text{ and } |x|+|y|<1),$$

where F_2 is Appell function [94,p.53(5)].

We thus find that

$$\Omega^{\mu}_{\eta,\nu,\lambda}[x,q,\gamma-(s+t)(\omega+1)+zt/s+y\omega]=\frac{x^{\nu+2\lambda+1}\gamma^{-\frac{1}{2}(\eta+\nu+2\lambda+1)}}{z^{\nu+2\lambda+3/2}}\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{(s/\gamma)^m(t/\gamma)^n}{m!\;n!}$$

$$\sum_{l,k=-\infty}^{\infty}\frac{(-x^2/4\gamma)^l(-q/\sqrt{\gamma})^k}{\overline{|\lambda+l+1}\;\overline{|\nu+\lambda+\mu l+1}}\sum_{p=0}^{\infty}\frac{\omega^p}{p!}(1+m+n-p)_p\,F_2[\tfrac{1}{2}(\eta+\nu+2\lambda+2m+2n+2l+k+1);-p,-n;$$

$$1+m+n-p,\ m+1;\ \frac{x}{\gamma},\ \frac{z}{\gamma}]\,\overline{|(\tfrac{1}{2}(\eta+\nu+2\lambda+2n+2m+2l+k+1))}. \tag{6.4.5}$$

In view of the result (cf.(4.2.8)):

$$\sum_{k=0}^{\infty}\frac{(-n-m)_k}{k!}(-x)^k\,F_2\,[\tfrac{1}{2}(\eta+\nu+2\lambda+2n+2m+2l+k+1);-n,-k;m+1,1+m+n-k;z,y]$$

$$=\quad F^{1:1;0}_{0:1;0}\left[\begin{matrix}\tfrac{1}{2}(\eta+2\lambda+\nu+2m+2n+2l+k+1):-n & ;—; \\ \rule{6cm}{0.4pt}: & m+1;—;\end{matrix}\ z,\ -xy\right]. \tag{6.4.6}$$

we get

$$\Omega^{\mu}_{\eta,\nu,\lambda}[x,q,\gamma-(s+t)(1+\omega)+zt/s+y\omega]=\frac{x^{\nu+2\lambda+1}\gamma^{-\frac{1}{2}(\eta+\nu+2\lambda+1)}}{2^{\nu+2\lambda+3/2}}$$

$$\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{(s/\gamma)^m(t/\gamma)^n}{n!m!}\sum_{l,k=0}^{\infty}\frac{(-x^2/4\gamma)^l(-q/\sqrt{\gamma})^k}{k!\overline{|\lambda+l+1}\,\overline{|\nu+\lambda+\mu l+1}}\,\overline{|\tfrac{1}{2}(\eta+\nu+2\lambda+2m+2n+2l+k+1)}$$

$$F^{1:1;0}_{0:1;0}\left[\begin{matrix}\tfrac{1}{2}(\eta+\nu+2\lambda+2m+2n+2l+k+1: & -n\ ;\ -; \\ \rule{6cm}{0.4pt}: & m+1;\ -;\end{matrix}\ \frac{z}{\gamma},\ \frac{-\omega y}{\gamma}\right]. \tag{6.4.7}$$

Now, separating the k-series into its even and odd terms, we get

$$\Omega^{\mu}_{\eta,v,\lambda}[x,q,\gamma-(s+t)(1+\omega)+zt/s+y\omega]= \frac{x^{v+2\lambda+1}\ \gamma^{-\frac{1}{2}(\eta+v+2\lambda+1)}}{2^{v+2\lambda+3/2}}$$

$$\sum_{m=-\infty}^{\infty}\ \sum_{n=m^*}^{\infty} \frac{(s/\gamma)^m\ (t/\gamma)^n}{n!m!} \sum_{l=0}^{\infty} \frac{(-x^2/4\gamma)^l}{\Gamma(\lambda+l+1)\,\Gamma(v+\lambda+\mu l+1)} \Big\{\Gamma(\tfrac{1}{2}(\eta+2\lambda+v+2m+2n+2l+1)$$

$$F^{(3)}\left[\begin{array}{c} \tfrac{1}{2}(\eta+2\lambda+v+2m+2n+2l+1::-;-;-:-n\quad ;-;-; \\ \hline ::-;-;-:m+1;-;\tfrac{1}{2}; \end{array}\ \frac{z}{\gamma}, \frac{-\omega y}{\gamma}, \frac{q^2}{4\gamma}\right]$$

$$-\frac{q}{\sqrt{\gamma}}\ \Gamma(\tfrac{1}{2}(\eta+v+2\lambda+2m+2n+2l+2)$$

$$F^{(3)}\left[\begin{array}{c} \tfrac{1}{2}(\eta+2\lambda+v+2m+2n+2l+2::\ -;\ -;-:-n\quad ;-;-; \\ \hline ::\ -;\ -;-:\ m+1;-;3/2; \end{array}\ \frac{z}{\gamma}, \frac{-\omega y}{\gamma}, \frac{q^2}{4\gamma}\right]\Big\} \quad (6.4.8)$$

When $s=t=\dfrac{z+y\omega}{2(1+\omega)}$, it is not difficult to observe that:

$$\Omega^{\mu}_{\eta,v,\lambda}[x,q,\gamma-(s+t)(1+\omega)+zt/s+y\omega]= \Omega^{\mu}_{\eta,v,\lambda}[x,q,\gamma] \qquad (6.4.9)$$

Further, if in (6.4.8), we set $\mu-1=\lambda=0$, $\gamma=\frac{1}{4}$, $s=t=\dfrac{z+y\omega}{2(\omega+1)}$, $\eta=\frac{1}{2}$ and $v=\pm\frac{1}{2}$, we shall obtain representations of the Voigt functions $K(x,y)$ and $L(x,y)$.

Finally if in (6.4.8), we set $y=\omega=0$, it reduces to a known result due to Srivastava et al. [96,p.59(3.6)].

6.5 GENERATING RELATIONS OBTAINABLE BY MEANS OF EXPLICIT REPRESENTATION OF $\Omega_{\mu,\nu}$ [x,y,z]

Klusch [52] introduced an explicit expression of the generalized Voigt function $\Omega_{\mu,\nu}$ $[x,q,\gamma]$ in the form

$$\Omega_{\mu,\nu}\,[x,q,\gamma]=\frac{x^{\nu+\frac{1}{2}}\,\gamma^{-\frac{1}{2}(\mu+\nu+1)}}{2^{\nu+3/2}\,\Gamma(\nu+1)}\left\{\Gamma\left(\tfrac{1}{2}(\mu+\nu+1)\right)\Psi_2\left[\tfrac{1}{2}(\mu+\nu+1);\nu+1,\tfrac{1}{2};\frac{-x^2}{4\gamma},\frac{q^2}{4\gamma}\right]\right.$$

$$\left.-\frac{q}{\sqrt{\gamma}}\,\Gamma\left(\tfrac{1}{2}(\mu+\nu+1)\right)\Psi_2\left[\tfrac{1}{2}(\mu+\nu+2);\nu+1,3/2;\frac{-x^2}{4\gamma},\frac{q^2}{4\gamma}\right]\right\} \tag{6.5.1}$$

where Ψ_2 denotes one of Humbert's confluent hypergeometric functions of two variables [94,p.59(42)]. On taking $\lambda=\mu-1=0$ in (6.4.8) and expanding the left member of the resulting expression by means of the representation (6.5.1), we get the following generating relation

$$\left(\frac{\gamma-(s+t)(1+\omega)+zt/s+y\omega}{\gamma}\right)^{-\alpha}\left\{\Gamma(\alpha)\,\Psi_2\left[\alpha;\nu+1,\tfrac{1}{2};\frac{-x^2}{4(\gamma-(s+t)(1+\omega)+zt/s+y\omega)},\frac{q^2}{4(\gamma-(s+t)(1+\omega)+zt/s+y\omega)}\right]\right.$$

$$\left.-\frac{q\,\Gamma(\alpha+\frac{1}{2})}{\sqrt{\gamma-(s+t)(1+\omega)+zt/s+y\omega}}\,\Psi_2\left[\alpha+\tfrac{1}{2};\nu+1,3/2;\frac{-x^2}{4(\gamma-(s+t)(1+\omega)+zt/s+y\omega)},\frac{q^2}{4(\gamma-(s+t)(1+\omega)+2zt/s+y\omega)}\right]\right\}$$

$$=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{(s/\gamma)^m\,(t/\gamma)^n}{m!\,n!}\left\{\Gamma(\alpha+m+n)\,F_P^{(4)}\left[\begin{matrix}\alpha+m+n::-;-;-;-:-;-;-;-:-n;-;-;-; \\ -::-;-;-;-:-;-;-;-:m+1;-;\nu+1;\frac{1}{2};\end{matrix}\ \frac{z}{\gamma},\frac{-\omega y}{\gamma},\frac{-x^2}{4\gamma},\frac{q^2}{4\gamma}\right]\right.$$

$$\left.-\frac{q\,\Gamma(\alpha+m+n+\frac{1}{2})}{\sqrt{\gamma}}\,F_P^{(4)}\left[\begin{matrix}\alpha+m+n+\frac{1}{2}::-;-;-;-:-;-;-;-:-n;-;-;-; \\ -::-;-;-;-:-;-;-;-:m+1;-;\nu+1;3/2;\end{matrix}\ \frac{z}{\gamma},\frac{-\omega y}{\gamma},\frac{-x^2}{4\gamma},\frac{q^2}{4\gamma}\right]\right\} \tag{6.5.2}$$

Some special cases of (6.5.2) are worthy of note. Indeed, for $y=\omega=0$, relation (6.5.2) reduces to result due to Srivastava et al. [96,p.62(4.1)]. For $q=0$, $\gamma=1$ and $x^2=-4u$, (6.5.2) evidently reduces to a generating function for $F^{(3)}$ in terms of ${}_1F_1$ and we thus obtain

$$\left(1-(s+t)(1+\omega)+zt/s+y\omega\right)^{-\alpha}\ {}_1F_1\left[\alpha;v+1;\ \frac{u}{(1-(s+t)(1+\omega)+zt/s+y\omega)}\right]$$

$$=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{s^m\ t^n}{m!n!}(\alpha)_{m+n}\ F^{(3)}\left[\begin{array}{l}\alpha+m+n::-;-;-:-n;-;-;\\ \text{———}::-;-;-:m+1;-;v+1;\end{array}\ z,-\omega y,u\right] \quad (6.5.3)$$

For $v=\alpha-1$, (6.5.3) reduces to

$$\left(1-(s+t)(1+\omega)+zt/s+y\omega\right)^{-\alpha}\ \exp\left(\frac{u}{(1-(s+t)(1+\omega)+zt/s+y\omega)}\right)$$

$$=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{s^m\ t^n}{m!n!}(\alpha)_{m+n}\ F^{(3)}\left[\begin{array}{l}\alpha+m+n::-;-;-:-n;-;-;\\ \text{———}::-;-;-:m+1;-;\alpha;\end{array}\ z,-\omega y,u\right] \quad (6.5.4)$$

On letting $u\to 0$, (6.5.3) reduces to

$$\left(1-(s+t)(1+\omega)+zt/s+y\omega\right)^{-\alpha}$$

$$=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{s^m\ t^n}{m!n!}(\alpha)_{m+n}\ F_{0:1;0}^{1:1;0}\left[\begin{array}{l}\alpha+m+n:-n;-;\\ \text{——}:m+1;-;\end{array}\ z,-\omega y\right] \quad (6.5.5)$$

Now, taking $\omega=0$ in (6.5.5), we obtain

$$\left(1-s-t+zt/s\right)^{-\alpha} = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{s^m t^n}{m!n!} (\alpha)_{m+n}\ {}_2F_1\ [\alpha+m+n,\ -n;\ m+1;z] \quad (6.5.6)$$

On other hand, if in (6.5.5), $z \to 0$, we get

$${}_1F_0\left[\alpha;—:(s+t)(\omega+1)+y\omega\right] = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{s^m\ t^n}{m!\ n!} (\alpha)_{m+n}\ {}_1F_0[\alpha+m+n;–;\ -\omega y] \quad (6.5.7)$$

If in (6.5.7), we replace s, t and y by s/u, t/u and y/u respectively, multiply both the sides by $u^{-\lambda}$ and then take the inverse Laplace transform [94,p.219(7)] (cf. (4.2.6)), we get

$${}_1F_1\left[\alpha;\lambda:(s+t)(\omega+1)+y\omega\right] = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{s^m\ t^n\ (\alpha)_{m+n}}{m!n!(\lambda)_{m+n}}\ {}_1F_1[\alpha+m+n;\lambda;\ -\omega y] \quad (6.5.8)$$

Further, if in (6.5.8), we replace s, t and y by su, tu and yu respectively, multiply both the sides by $u^{\sigma-1}\ e^{-u}$ and take Laplace transform [94, p.219(6)], (cf. (4.2.3)), equation (6.5.8) yields

$${}_2F_1\left[\alpha,\sigma;\lambda;(s+t)(\omega+1)+y\omega\right] = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{s^m\ t^n}{m!n!} \frac{(\alpha)_{m+n}(\sigma)_{m+n}}{(\lambda)_{m+n}}$$

$${}_2F_1[\alpha+m+n,\ \sigma;\lambda;\ -\omega y] \quad (6.5.9)$$

For $\omega, y \to 0$, equation (6.5.3) evidently reduces to

$$\left(1-s-t+zt/s\right)^{-\alpha}\ {}_1F_1\left[\alpha;v+1;\frac{u}{(1-s-t+zt/s)}\right]$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{s^m\ t^n}{m!n!} (\alpha)_{m+n}\ \Psi_1[\alpha+m+n,\ -n;m+1,v+1;z,u] \quad (6.5.10)$$

where Ψ_1 is a confluent hypergeometric function of two variables [94, p.59(41)].

When $z,\omega \to 0$, in (6.5.3), we obtain

$$(1-s-t)^{-\alpha} \; {}_1F_1\left[\alpha;v+1;\frac{u}{(1-s-t)}\right]$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{s^m \; t^n}{m! \; n!} (\alpha)_{m+n} \; {}_1F_1 \left[\alpha+m+n;v+1;u\right]. \tag{6.5.11}$$

CHAPTER -VII

ON TRANSFORMATION AND REDUCTION FORMULAE OF CERTAIN HYPERGEOMETRIC FUNCTIONS

7.1 INTRODUCTION

The aim of this chapter is to obtain a number of transformation and reduction formulae involving hypergeometric functions of two and three variables. An integral formula involving the product of three Bessel functions of different order and arguments considered in the present chapter plays a key role in obtaining the main transformation formulae (7.3.1) and (7.3.2), for from it easily follows transformations, reductions and deductions involving Kampe' de Fe'riet function $F^{A:B;B'}_{C:D;D'}[x,y]$, Srivastava function $F^{(3)}[x,y,z]$, Appell function F_4, Horn functions H_4 and $H_4^{(p)}$ and Lauricella functions $F_A^{(3)}$ and $F_C^{(3)}$.

Transformation formula for one and more variables have been given considerable attention in the literature, see for example, Agrawal [2], Prudnikov et al. [71], Srivastava and Karlsson [93], Exton [27, Chapter 4] and Srivastava and Manocha [94, p. 95-98]. Prudnikov et al. [71] provides an impressive tabulation and a wealth of information on series and transformations. Although various transformation and reduction formulae of functions of two and three variables are known, we hope that our results are also worth attention for one reason that some of our formulae have somthing of the simplicity because they typically involve variables which are not equal or opposite. In particular some special cases of our results e.g. (7.3.5) to (7.3.8) and (7.3.11) to (7.3.17) appear to be new.

The following reduction formulae

$$F^{(3)} \to F_C^{(3)},\ H_4^{(p)},\ F_A^{(3)},\ F_4,\ F_E,\ H_4,$$

$$F^{2:1;0}_{0:3;1} \to \Psi_2\ ,\ {}_0F_1\ ,$$

$$F^{2:2:0}_{0:3:1} \to F_C^{(3)}\ ,\ F_A^{(3)}\ ,$$

and

$$F_C^{(3)} \to F_4\ ,\ ({}_2F_1 + {}_2F_1)\ ,$$

are an immediate consequence of section 7.2. Also known results due to Pathan and Khan [66], Erde'lyi [22] and Watson [98] are special cases of our results of this chapter.

7.2 INTEGRALS

In order to obtain the main transformations of this chapter, we establish an integral in the form

$$I^{\sigma,\delta,v}_{(z,y,x)} = \int_0^\infty u^\eta \exp(-qu-\gamma u^2)\ J_\sigma(zu)\ J_\delta(yu)\ J_v(xu)\ du\ , \tag{7.2.1}$$

$$= z^\sigma\ y^\delta\ x^v\ K\ \Big\{\ \Gamma(\tfrac{1}{2}(\eta+\sigma+\delta+v+1))$$

$$\Psi_2^{(4)}\Big[\tfrac{1}{2}(\eta+\sigma+\delta+v+1);\sigma+1,\delta+1,v+1,\ \tfrac{1}{2};\ \frac{-z^2}{4\gamma},\ \frac{-y^2}{4\gamma},\ \frac{-x^2}{4\gamma},\ \frac{q^2}{4\gamma}\ \Big]$$

$$-\ \frac{q}{\sqrt{\gamma}}\ \Gamma(\tfrac{1}{2}(\eta+\sigma+\delta+v+2))$$

$$\Psi_2^{(4)}\Big[\tfrac{1}{2}(\eta+\sigma+\delta+v+2);\sigma+1,\delta+1,v+1,\ 3/2;\ \frac{-z^2}{4\gamma},\ \frac{-y^2}{4\gamma},\ \frac{-x^2}{4\gamma},\ \frac{q^2}{4\gamma}\ \Big]\ \Big\}, \tag{7.2.2}$$

$$= z^{\sigma}y^{\delta}x^{v}\, K \left(1-\frac{y^2}{z^2}\right)^{\sigma+\delta+1} \Big\{ \Gamma(\tfrac{1}{2}(\eta+\sigma+\delta+v+1)) \sum_{n=0}^{\infty} \frac{(\sigma+\delta+1)_n}{n!} \left(\frac{y^2}{z^2}\right)^n$$

$$F^{(3)}\left[\begin{matrix} \tfrac{1}{2}(\eta+\sigma+\delta+v+1)::-;-;-:\sigma+\delta+n+1,\delta+n+1;—;—; \\ ——::-;-;-:\sigma+\delta+1,\sigma+1,\delta+1;v+1;\tfrac{1}{2}; \end{matrix} \frac{-z^2}{4\gamma}\left(1-\frac{y^2}{z^2}\right)^2, \frac{-x^2}{4\gamma}, \frac{q^2}{4\gamma}\right]$$

$$- \frac{q}{\sqrt{\gamma}}\, \Gamma(\tfrac{1}{2}(\eta+\sigma+\delta+v+2)) \sum_{n=0}^{\infty} \frac{(\sigma+\delta+1)_n}{n!} \left(\frac{y^2}{z^2}\right)^n$$

$$F^{(3)}\left[\begin{matrix} \tfrac{1}{2}(\eta+\sigma+\delta+v+2)::-;-;-:\sigma+\delta+n+1,\delta+n+1;-;-; \\ ——::-;-;-:\sigma+\delta+1,\sigma+1,\delta+1;v+1;3/2; \end{matrix} \frac{-z^2}{4\gamma}\left(1-\frac{y^2}{z^2}\right)^2, \frac{-x^2}{4\gamma}, \frac{q^2}{4\gamma}\right]\Big\}, \quad (7.2.3)$$

$$I^{\sigma,\delta,v}_{(y,y,x)} = y^{\sigma+\delta}\, x^{v} \Big\{ \Gamma(\tfrac{1}{2}(\eta+\sigma+\delta+v+1))$$

$$F^{(3)}\left[\begin{matrix} \tfrac{1}{2}(\eta+\sigma+\delta+v+1)::-;-;-:\tfrac{1}{2}(\sigma+\delta+2),\tfrac{1}{2}(\sigma+\delta+1);—;-; \\ ——::-;-;-:\sigma+1,\delta+1,\sigma+\delta+1 \quad ;v+1;\tfrac{1}{2}; \end{matrix} \frac{-y^2}{\gamma}, \frac{-x^2}{4\gamma}, \frac{q^2}{4\gamma}\right]$$

$$- \frac{q}{\sqrt{\gamma}}\, \Gamma(\tfrac{1}{2}(\eta+\sigma+\delta+v+2))$$

$$F^{(3)}\left[\begin{matrix} \tfrac{1}{2}(\eta+\sigma+\delta+v+2)::-;-;-:\tfrac{1}{2}(\sigma+\delta+2),\tfrac{1}{2}(\sigma+\delta+1);—;-; \\ ——::-;-;-:\sigma+1,\delta+1,\sigma+\delta+1 \quad ;v+1;3/2; \end{matrix} \frac{-y^2}{\gamma}, \frac{-x^2}{4\gamma}, \frac{q^2}{4\gamma}\right]\Big\}, \quad (7.2.4)$$

where $\Psi_2^{(n)}$ is a confluent series of n-variables [94,p.34(9)] given in equation (1.10.7), $J_v(z)$ is Bessel polynomials defined by [73,p.110(1)]

$$J_v(z) = \sum_{m=0}^{\infty} \frac{(-1)^m\,(z/2)^{v+2m}}{m!\,\Gamma(v+n+1)}, \quad |z|<\infty, \quad (7.2.5)$$

and

$$K = \frac{\gamma^{-\frac{1}{2}(\eta+\sigma+\delta+v+1}}{\Gamma(\sigma+1)\ \Gamma(\delta+1)\ \Gamma(v+1)\ 2^{\sigma+\delta+v+1}} \ .$$

<u>**Derivation of formulae (7.2.2) to (7.2.4)**</u>

To establish (7.2.2), expressing $J_\sigma(z,u)$, $J_\delta(yu)$ and $J_v(xu)$ in series and expanding exp(-qu) in the form

$$\sum_{k=0}^{\infty} \frac{(-qu)^k}{k!} \ , \tag{7.2.6}$$

integrating term by term with the help of the result (cf. [13,p.58(3.4)]).

$$\int_0^{\infty} x^{s-1} \exp(-\alpha x^2)\, dx = \tfrac{1}{2}\, \alpha^{-\frac{1}{2}s}\ \Gamma(\tfrac{1}{2}s) \ , \tag{7.2.7}$$

$$(\ \mathrm{Re}(s)>0\ ,\ \mathrm{Re}(x)>0\ ),$$

and then separating the k-series into its even and odd terms [94,p.200(3)], we arrive at (7.2.2). If we use the relation (cf. [23,p.11(47) and p.64(23)])

$$\Gamma(\sigma+1)\ \Gamma(\delta+1)\ J_\sigma(zu)\ J_\delta(yu)$$

$$= \left(\frac{zu}{2}\right)^{\sigma} \left(\frac{yu}{2}\right)^{\delta} \sum_{m=0}^{\infty} \frac{(-1)^m\ (zu/2)^{2m}}{m!\ (\sigma+1)_m} \left(1-\frac{y^2}{z^2}\right)^{\sigma+\delta+2m+1} {}_2F_1\left[\begin{matrix} \delta+m+1,\ \sigma+\delta+m+1; \\ \\ \delta+1; \end{matrix}\ \ y^2/z^2\right], \tag{7.2.8}$$

replace $J_v(xu)$ by its series representation, expand exp(-qu) as in (7.2.6), integrate term by term with the help of the result (7.2.7) and then separate the k-series into its even and odd terms, we get (7.2.3). On the other hand, if in (7.2.1), we set z=y, apply the relation [23,p.11(44)]

$$J_\sigma(yu)\,J_\delta(yu) = \frac{(xu/2)^{\sigma+\delta}}{\Gamma(\sigma+1)\,\Gamma(\delta+1)}\;{}_2F_3\left[\begin{matrix}\tfrac{1}{2}(\sigma+\delta+1),\ \tfrac{1}{2}(\sigma+\delta+2); \\ \\ \sigma+1,\ \delta+1,\ \sigma+\delta+1;\end{matrix}\ -y^2u^2\right], \quad (7.2.9)$$

express $J_v(xu)$, expand exp(-qu) as in (7.2.6) and then separate the k-series into its even and odd terms, we arrive at (7.2.4).

7.3 TRANSFORMATION, REDUCTION AND DEDUCTION FORMULAE

If we compare (7.2.2), (7.2.3) and (7.2.4) and adjust the parameters, we get the following transformations :

$$\left(1-\frac{y}{z}\right)^{\sigma+\delta+1}\left\{\Gamma\!\left(\tfrac{1}{2}(\eta+\sigma+\delta+v+1)\right)\sum_{n=0}^{\infty}\frac{(\sigma+\delta+1)_n}{n!}\left(\frac{y}{z}\right)^n\right.$$

$$F^{(3)}\left[\begin{matrix}\tfrac{1}{2}(\eta+\sigma+\delta+v+1)::-;-;-:\sigma+\delta+n+1,\delta+n+1;\text{—};\text{—}; \\ \\ \text{——}::-;-;-:\sigma+\delta+1,\sigma+1,\delta+1;v+1;\ \tfrac{1}{2};\end{matrix}\ z\left(1-\frac{y}{z}\right)^2,\ x\ ,\ q\right]$$

$$-2\sqrt{q}\ \Gamma\!\left(\tfrac{1}{2}(\eta+\sigma+\delta+v+2)\right)\sum_{n=0}^{\infty}\frac{(\sigma+\delta+1)_n}{n!}\left(\frac{y}{z}\right)^n$$

$$\left.F^{(3)}\left[\begin{matrix}\tfrac{1}{2}(\eta+\sigma+\delta+v+2)::-;-;-:\sigma+\delta+n+1,\delta+n+1;\text{—};\text{—}; \\ \\ \text{——}::-;-;-:\sigma+\delta+1,\sigma+1,\delta+1;v+1;\tfrac{3}{2};\end{matrix}\ z\left(1-\frac{y}{z}\right)^2,\ x,\ q\right]\right\}$$

$$= \Big\{ \Gamma(\tfrac{1}{2}(\eta+\sigma+\delta+v+1))\ \Psi_2^{(4)}\ \big[\tfrac{1}{2}(\eta+\sigma+\delta+v+1);\sigma+1;\delta+1,v+1,\tfrac{1}{2};\ z,y,x,q\big]$$

$$-2\sqrt{q}\,\Gamma(\tfrac{1}{2}(\eta+\sigma+\delta+v+1))\ \Psi_2^{(4)}\ \big[\tfrac{1}{2}(\eta+\sigma+\delta+v+1);\sigma+1;\delta+1,v+1,3/2;\ z,y,x,q\big]\Big\}, \quad (7.3.1)$$

$$\Big\{\Gamma(\tfrac{1}{2}(\eta+\sigma+\delta+v+1))\ F^{(3)}\left[\begin{matrix}\tfrac{1}{2}(\eta+\sigma+\delta+v+1)::-;-;-:\tfrac{1}{2}(\sigma+\delta+2),\tfrac{1}{2}(\sigma+\delta+1);-;-; \\ \text{———}::-;-;-:\sigma+1,\delta+1,\sigma+\delta+1;v+1;\tfrac{1}{2};\end{matrix}\ 4y,x,q\right]$$

$$-2\sqrt{q}\,\Gamma(\tfrac{1}{2}(\eta+\sigma+\delta+v+2))\ F^{(3)}\left[\begin{matrix}\tfrac{1}{2}(\eta+\sigma+\delta+v+2)::-;-;-:\tfrac{1}{2}(\sigma+\delta+2),\tfrac{1}{2}(\sigma+\delta+1);-;-; \\ \text{———}::-;-;-:\sigma+1,\delta+1,\sigma+\delta+1;v+1;3/2;\end{matrix}\ 4y,x,q\right]\Big\}$$

$$= \{\Gamma(\tfrac{1}{2}(\eta+\sigma+\delta+v+1))\ \psi_2^{(4)}\ [\tfrac{1}{2}(\eta+\sigma+\delta+v+1);\ \sigma+1;\delta+1,v+1,\tfrac{1}{2};\ y,\ y,\ x,\ q\ ]$$

$$-2\sqrt{q}\,\Gamma(\tfrac{1}{2}(\eta+\sigma+\delta+v+2))\ \psi_2^{(4)}\ [\tfrac{1}{2}(\eta+\sigma+\delta+v+2);\ \sigma+1;\delta+1,v+1,3/2;\ y,\ y,\ x,\ q\ ]\}. \quad (7.3.2)$$

In (7.3.1) and (7.3.2) on replacing z, y, x and q by zt, yt, xt and qt respectively, multiplying both the sides by $t^{\beta-1}\ e^{-t}$ and taking the Laplace transform with the help of the result [28,p.225(A.6.2.6)]:

$$F_c^{(n)}\ [a,p;\ c_1,\dots,c_n;\ x_1,\dots,x_n] = \frac{1}{\Gamma p}\int_0^\infty e^{-t}\ t^{p-1}\ \Psi_2^{(n)}\ [a;\ c_1,\dots,c_n;\ x_1 t,\dots,x_n t]dt, \quad (7.3.3)$$

we get

$$\left(1-\frac{y}{z}\right)^{\sigma+\delta+1}\ \{\Gamma(\tfrac{1}{2}(\eta+\sigma+\delta+v+1))\sum_{n=0}^{\infty}\frac{(\sigma+\delta+1)_n}{n!}\left(\frac{y}{z}\right)^n \Gamma\beta$$

$$F^{(3)}\left[\begin{matrix}\tfrac{1}{2}(\eta+\sigma+\delta+v+1),\beta::-;-;-:\sigma+\delta+n+1,\delta+n+1;-;-; \\ \text{———}::-;-;-:\sigma+\delta+1,\sigma+1,\delta+1;v+1;\tfrac{1}{2};\end{matrix}\ z\left(1-\frac{y}{z}\right)^2,\ x,\ q\right]$$

$$-2\sqrt{q}\ \Big\{ \Gamma(\tfrac{1}{2}(\eta+\sigma+\delta+v+2)) \sum_{n=0}^{\infty} \frac{(\sigma+\delta+1)_n}{n!} \left(\frac{y}{z}\right)^n \Gamma\beta+\tfrac{1}{2}$$

$$F^{(3)}\left[\begin{matrix} \tfrac{1}{2}(\eta+\sigma+\delta+v+2),\beta+\tfrac{1}{2}::-;-;-:\sigma+\delta+n+1,\delta+n+1;-;-; \\ \qquad\qquad\qquad\qquad\qquad\qquad z\left(1-\frac{y}{z}\right)^2, x, q \\ \text{———}::-;-;-:\sigma+\delta+1,\sigma+1,\delta+1;v+1;3/2; \end{matrix}\right]\Big\}$$

$$= \Big\{ \Gamma(\tfrac{1}{2}(\eta+\sigma+\delta+v+1))\ \Gamma\beta\ F_C^{(4)}\ [\tfrac{1}{2}(\eta+\sigma+\delta+v+1),\beta;\ \sigma+1,\delta+1,v+1,\tfrac{1}{2};\ z,\ y,\ x,\ q\ ]$$

$$-2\sqrt{q}\,\Gamma(\tfrac{1}{2}(\eta+\sigma+\delta+v+2))\ \Gamma\beta+\tfrac{1}{2}\ F_C^{(4)}\ [\tfrac{1}{2}(\eta+\sigma+\delta+v+2),\beta+\tfrac{1}{2};\sigma+1,\delta+1,v+1,3/2;\ z,y,x,q\ ]\ \Big\},$$

(7.3.4)

and

$$\Big\{ \Gamma(\tfrac{1}{2}(\eta+\sigma+\delta+v+1))\,\Gamma\beta\ F^{(3)}\left[\begin{matrix} \tfrac{1}{2}(\eta+\sigma+\delta+v+1),\beta::-;-;-:\tfrac{1}{2}(\sigma+\delta+2),\tfrac{1}{2}(\sigma+\delta+1);-;-; \\ \qquad\qquad\qquad\qquad 4y,x,q \\ \text{———}::-;-;-:\sigma+\delta+1,\sigma+1,\delta+1;v+1;\tfrac{1}{2}; \end{matrix}\right]$$

$$-2\sqrt{q}\,\Gamma(\tfrac{1}{2}(\eta+\sigma+\delta+v+2))\,\Gamma\beta+\tfrac{1}{2}\ F^{(3)}\left[\begin{matrix} \tfrac{1}{2}(\eta+\sigma+\delta+v+2),\beta+\tfrac{1}{2}::-;-;-:\tfrac{1}{2}(\sigma+\delta+2),\tfrac{1}{2}(\sigma+\delta+1);-;-; \\ \qquad\qquad\qquad\qquad 4y,x,q \\ \text{———}::-;-;-:\sigma+\delta+1,\sigma+1,\delta+1;v+1;3/2; \end{matrix}\right]\Big\}$$

$$= \Big\{ \Gamma(\tfrac{1}{2}(\eta+\sigma+\delta+v+1))\,\Gamma\beta\ F_C^{(4)}\ [\tfrac{1}{2}(\eta+\sigma+\delta+v+1),\beta;\ \sigma+1,\delta+1,v+1,\tfrac{1}{2};\ y,\ y,\ x,\ q\ ]$$

$$-2\sqrt{q}\,\Gamma(\tfrac{1}{2}(\eta+\sigma+\delta+v+2))\ \Gamma\beta+\tfrac{1}{2}\ F_C^{(4)}\ [\tfrac{1}{2}(\eta+\sigma+\delta+v+2),\beta+\tfrac{1}{2};\sigma+1,\delta+1,v+1,3/2;\ y,y,x,q\ ]\Big\}, \quad (7.3.5)$$

where $F_C^{(n)}$ is Lauricella's hypergeometric function of n-variables [9,p.60], defined by (1.10.3).

When $q \to 0$, formula (7.3.4) reduces to

$$\left(1-\frac{y}{z}\right)^{\sigma+\delta+1} F^{(3)}\left[\begin{array}{l} -::\sigma+\delta+1,\delta+1;-;\tfrac{1}{2}(\eta+\sigma+\delta+1),\beta:\text{———};\text{—};\text{—}; \\ \qquad\qquad\qquad\qquad\qquad\qquad z\left(1-\frac{y}{z}\right)^2,\ \frac{y}{z},\ x \\ -::\text{———};-;\text{———}:\sigma+\delta+1,\sigma+1,\delta+1;\delta+1;v+1; \end{array}\right]$$

$$= F_C^{(3)}\,[\tfrac{1}{2}(\eta+\sigma+\delta+v+1),\ \beta;\ \sigma+1,\ \delta+1,\ v+1;\ z\ ,\ y\ ,\ x\ ]. \qquad (7.3.6)$$

Now, if in (7.3.6), we set $2\beta=\eta+\sigma+\delta+v=\alpha$, replace z, y and x by z^2, y^2 and x^2 respectively and apply the result [93,p.331(225)],

$$F_c^{(n)}\,[\tfrac{1}{2}a,\ \tfrac{1}{2}a+\tfrac{1}{2};\ c_1,\ldots,c_n;\ x_1^2,\ldots,x_n^2] = (1-x_1+\ldots\ldots+x_n)^{-a}$$

$$F_A^{(n)}\,[a,c_1-\tfrac{1}{2},\ldots,\ c_n-\tfrac{1}{2};\ 2c_1-1,\ldots,2c_n-1;\ 2x_1/(1+x_1+\ldots+x_n),\ldots\ldots,\ 2x_n/(1+x_1+\ldots x_n)], \qquad (7.3.7)$$

$$\sum_{j=1}^{n}\ |\ x_j\ /\ (1+x_1+\ldots+x_n)\ |\ <\ \tfrac{1}{2}\ ,$$

we get

$$\left(1-\frac{y}{z}\right)^{\sigma+\delta+1} F^{(3)}\left[\begin{array}{l} -::\sigma+\delta+1,\delta+1;-;\tfrac{1}{2}\alpha+\tfrac{1}{2},\tfrac{1}{2}\alpha:\text{———};\text{—};\text{—}; \\ \qquad\qquad\qquad\qquad\qquad\qquad z^2\left(1-\frac{y^2}{z^2}\right)^2,\ \frac{y^2}{z^2},\ x^2 \\ -::\text{———};-;\text{———}:\sigma+\delta+1,\sigma+1,\delta+1;\delta+1;v+1; \end{array}\right]$$

$$=(1+z+y+x)^{-\alpha}\ F_A^{(3)}\left[\alpha,\sigma+\tfrac{1}{2},\delta+\tfrac{1}{2},v+\tfrac{1}{2};2\sigma+1,2\delta+1,2v+1;\ \frac{2z}{1+z+y+x},\ \frac{2y}{1+z+y+x},\ \frac{2x}{1+z+y+x}\right]. \qquad (7.3.8)$$

Further, on setting $v=\delta$, $\beta=\delta+1$, $\eta=\sigma+1$ in (7.3.6) and applying the result [93,p.329(218)]

$$F_c^{(3)} \; [a+b+1,\ b+1;\ a+1,\ b+1,\ b+1;\ x,\ y,\ z\ ] = (1+x-y-z)^{-a-b-1}$$

$$F_4\ [\tfrac{1}{2}\ (a+b+1),\ \tfrac{1}{2}\ (a+b+2);\ a+1,\ b+1;\ X,\ Y]\ , \qquad (7.3.9)$$

where $X = \dfrac{4z}{(1+z-y-x)^2}$, $Y = \dfrac{4yx}{(1+z-y-x)^2}$,

we obtain

$$\left(1-\frac{y}{z}\right)^{\sigma+\delta+1} F^{(3)}\left[\begin{matrix} -::\sigma+\delta+1,\delta+1;-;\sigma+\delta+1,\delta+1:\text{———};\text{—};\text{—}; \\ -::\text{———};-;\text{———}:\sigma+\delta+1,\sigma+1;\delta+1;\delta+1; \end{matrix}\ z\left(1-\frac{y}{z}\right)^2, \frac{y}{z}, x\right]$$

$$= (1+z-y-x)^{-(\sigma+\delta+1)}\ F_4\ [\tfrac{1}{2}(\sigma+\delta+1),\tfrac{1}{2}(\sigma+\delta+2);\sigma+1,\delta+1;\ X,\ Y\ ]\ , \qquad (7.3.10)$$

When q, $x \to 0$ in (7.3.1), we get

$$\left(1-\frac{y}{z}\right)^{\sigma+\delta+1} F\,{}^{2:1;0}_{0:3;1}\left[\begin{matrix} \sigma+\delta+1,\delta+1:\ \tfrac{1}{2}(\eta+\sigma+\delta+v+1);\text{———}; \\ \text{———}:\ \sigma+\delta+1,\sigma+1,\delta+1;\delta+1; \end{matrix}\ z\left(1-\frac{y}{z}\right)^2, \frac{y}{z}\right]$$

$$= \Psi_2\ [\tfrac{1}{2}(\eta+\sigma+\delta+v+1);\ \sigma+1,\ \delta+1;\ z,\ y\ ]. \qquad (7.3.11)$$

On multiplying equation (7.3.11) by $t^{-\frac{1}{2}(\eta+\sigma+\delta+v+1)}e^t$, replacing z and y by z/t and y/t respectively, using Hankel's contour integral [72,p.32(1.5.1.5)]

$$1/(c)_n = \Gamma c \,/\, 2\pi i \quad \int e^t\, t^{-c-n}\, dt \quad , \qquad (7.3.12)$$

where n is non-negative integer and c does not take non-positive integer values and adjust the parameters, we get a known result due to Pathan and Khan [66,p.181(16)].

$${}_0F_1[-;\ a;\ x]\ {}_0F_1[-;\ t;\ y]$$

$$= \left(1-\frac{y}{x}\right)^{a+b-1} F^{2:1;0}_{0:3;1}\left[\begin{matrix} b,\ a+b-1: \text{——}, \text{——}; \text{—}; \\ \\ \text{————}: a, b, a+b-1;\ b; \end{matrix}\quad x\left(1-\frac{y}{x}\right)^2, \frac{y}{x}\right].$$

On the other hand, if we multiply equation (7.3.11) by

$$t^{\beta-1}\, e^{-t}\ {}_1F_1[\alpha;\ \gamma;\ \omega t], \qquad (7.3.13)$$

and apply the result [93,p.320(172)]

$F_E\,[a,a,a,b_1,b_2,b_2;\ c_1,c_2,c_3;\ x,\ y,\ z]$

$$= 1/\Gamma a \int_0^\infty e^{-t}\, t^{a-1}\ {}_1F_1[b_1;\ c_1;\ xt]\ \Psi_2[b_2;\ c_2,\ c_3;\ yt,\ zt]\ dt,$$

$$\mathrm{Re}\,(x+yz+2\sqrt{yz}) < 1,\ \mathrm{Re}\,(a)>0\ ,$$

we obtain a transformation formula involving the hypergeometric function of three variables F_E [94,p.66(26)] given by equation (1.13.9),

$$\left(1-\frac{y}{z}\right)^{\sigma+\delta+1} F^{(3)}\left[\begin{matrix} -::\beta;\sigma+\delta+1,\delta+1;-:\alpha;\tfrac{1}{2}(\eta+\sigma+\delta+1);-; \\ \\ -::\text{————};-;-:\gamma;\sigma+\delta+1,\sigma+1,\delta+1;\delta+1; \end{matrix}\quad \omega,\ z\left(1-\frac{y}{z}\right)^2, \frac{y}{z}\right]$$

$$= F_E\left[\beta,\beta,\beta,\alpha,\tfrac{1}{2}(\eta+\sigma+\delta+v+1),\ \tfrac{1}{2}(\eta+\sigma+\delta+v+1);\ \gamma,\sigma+1,\delta+1;\ \omega,z,y\right]. \qquad (7.3.14)$$

In (7.3.11), if we replace η and δ by $1+\sigma-\delta-v$ and σ respectively and compare with the result [93,p.322(182)]

$$\Psi_2[a;\ a,\ a;\ x,\ y] = e^{x+y}\ {}_0F_1[-;\ a;\ xy], \qquad (7.3.15)$$

we get

$${}_0F_1\,[\text{—};\ \sigma+1;\ zy]$$

$$=e^{-(z+y)}\left(1-\frac{y}{z}\right)^{2\sigma+1} F^{2:1;0}_{0:3;1}\left[\begin{matrix}2\sigma+1,\sigma+1:\sigma+1\ ,\text{—},\text{——};\text{——};\\ \text{————}:2\sigma+1,\sigma+1,\sigma+1;\sigma+1;\end{matrix}\ z\left(1-\frac{y}{z}\right)^2,\ \frac{y}{z}\right]. \tag{7.3.16}$$

If in (7.3.14), we set $\eta=1-v,\delta=\sigma$ and compare with the result [67,p.169(4.8)], we get a transformation of Srivastava's triple hypergeometric function $F^{(3)}[x,y,z]$ in terms of Horn's function H_4 [22,p.225(16)]

$$(1-z-y)^{-\beta}\ H_4\left[\beta,\alpha;\sigma+1,\gamma;\ \frac{2z}{(1-z-y)^2},\ \frac{\omega}{(1-z-y)}\right]$$

$$=\left(1-\frac{y}{z}\right)^{2\sigma+1} F^{(3)}\left[\begin{matrix}-::2\sigma+1,\sigma+1;-;\beta:\ \sigma+1,\text{——},-,\text{—};\text{——};\alpha;\\ -::\text{————};-;-:2\sigma+1,\sigma+1,\sigma+1;\sigma+1;\gamma;\end{matrix}\ z\left(1-\frac{y}{z}\right)^2,\ \frac{y}{z},\ \omega\right]. \tag{7.3.17}$$

When $q\to 0$ in (7.3.4), we get

$$F^{2:2;0}_{0:3;1}\left[\begin{matrix}\tfrac{1}{2}(\eta+\sigma+\delta+v+1),\beta:\tfrac{1}{2}(\sigma+\delta+2),\ \tfrac{1}{2}(\sigma+\delta+1);\text{———};\\ \text{——————}:\sigma+\delta+1,\sigma+1,\delta+1\quad ;v+1;\end{matrix}\ 4y,\ x\right]$$

$$=F_C^{(3)}\left[\tfrac{1}{2}(\eta+\sigma+\delta+v+1),\ \beta;\ \sigma+1,\ \delta+1,\ v+1;\ y,\ y,\ x\right]. \tag{7.3.18}$$

On setting $\sigma=a-1$, $\delta=b-1$, $v=b-1$, $\beta=b$, $\eta=a$ in (7.3.18), and comparing the resulting expression with results of Pathan and Khan [66,p.181(17), p.182(22) and p.183(23)], we get the following reduction formulae:

$$F_C^{(3)}\left[b,\ a+b-1;\ a,\ b,\ b;\ \frac{x}{4}\left(1-\frac{y}{x}\right)^2,\ \frac{x}{4}\left(1-\frac{y}{x}\right)^2,\ \frac{y}{x}\right]$$

$$=\left(1-\frac{y}{x}\right)^{-(a+b-1)} F_4[\tfrac{1}{2}(a+b-1),\ \tfrac{1}{2}(a+b);\ a,\ b;\ x,\ y], \qquad (7.3.19)$$

$$F_C^{(3)}[b,\ a+b-1;\ a,\ b,\ b;\ y,\ y,\ -1]$$

$$= 2^{-(a+b-1)}\ F_4\left[\tfrac{1}{2}(a+b-1),\ \tfrac{1}{2}(a+b);\ a,\ b;\ y,\ -y\right], \qquad (7.3.20)$$

$$F_C^{(3)}\left[b,\ a-\tfrac{1}{2};\ a,\ \tfrac{1}{2},\ \tfrac{1}{2};\ \frac{x}{4}\left(1-\frac{y}{x}\right)^2,\ \frac{x}{4}\left(1-\frac{y}{x}\right)^2,\ \frac{y}{x}\right]$$

$$=\left(1-\frac{y}{x}\right)^{-(a+\frac{1}{2})}\Big\{\tfrac{1}{2}(1+\sqrt{y})^{-(a+b-1)}\ {}_2F_1\ [\tfrac{1}{2}(a+b-1),\ \tfrac{1}{2}(a+b);\ a;\ x(1+\sqrt{y})^{-2}]$$

$$+\tfrac{1}{2}(1-\sqrt{y})^{-(a+b-1)}\ {}_2F_1\ [\tfrac{1}{2}(a+b-1),\ \tfrac{1}{2}(a+b);\ a\ ;\ x(1-\sqrt{y})^{-2}\ ]\Big\}. \qquad (7.3.21)$$

On letting $q\to 0$ in (7.3.2), replacing y and x by y/t and x/t respectively and multiplying both the sides by

$$t^{-\frac{1}{2}(\eta+\sigma+\delta+v+1)}\ e^{-t},$$

then evaluating the result obtained with the help of (7.3.12) and adjusting the parameters, we get

$${}_0F_1\ [-;\ a;\ y]\ {}_0F_1\ [-;\ b;y] = {}_3F_3\left[\begin{matrix}\tfrac{1}{2}(a+b-1),\ \tfrac{1}{2}(a+b); \\ a,\ b,\ a+b-1;\end{matrix}\ 4y\right]. \qquad (7.3.22)$$

Equation (7.3.22) is a known result of Erde'lyi [22,p.185(2)], which incidentally is equivalent to a well-known result of Watson [98,p.147(1)].

In (7.3.2), let $y \to 0$. Then we get

$$\Big\{ \Gamma(\tfrac{1}{2}(\eta+\sigma+\delta+v+1)) \; F^{1:0;0}_{0:1;1}\left[\begin{array}{l} \tfrac{1}{2}(\eta+\sigma+\delta+v+1) : \text{---};\text{---}; \\ \overline{\qquad\qquad\qquad} : v+1;\ \tfrac{1}{2}; \end{array} \; x,\ q\right]$$

$$-2\sqrt{q}\; \Gamma(\tfrac{1}{2}(\eta+\sigma+\delta+v+2)) \; F^{1:0;0}_{0:1;1}\left[\begin{array}{l} \tfrac{1}{2}(\eta+\sigma+\delta+v+2) : \text{---};\text{---}; \\ \overline{\qquad\qquad\qquad} : v+1;{}^{3}/_{2}; \end{array} \; x,\ q\right] \Big\}$$

$$= \Big\{ \Gamma(\tfrac{1}{2}(\eta+\sigma+\delta+v+1)) \quad \Psi_2\ [\tfrac{1}{2}(\eta+\sigma+\delta+v+1);\ v+1,\ \tfrac{1}{2};\ x,\ q\ ]$$

$$-\ 2\sqrt{q}\; \Gamma(\tfrac{1}{2}(\eta+\sigma+\delta+v+1)) \quad \Psi_2\ [\tfrac{1}{2}(\eta+\sigma+\delta+v+2);\ v+1,\ {}^{3}/2;\ x,\ q\ ]\ \Big\}\ . \qquad (7.3.23)$$

On replacing x by xt in (7.3.23), multiplying both the sides by $t^{\beta-1}$ and using the Laplace transforms [28,p.224(A.1.1.8)], we obtain

$$\Big\{ \Gamma(\tfrac{1}{2}(\eta+\sigma+\delta+v+1)) \quad \Psi_1\ [\tfrac{1}{2}(\eta+\sigma+\delta+v+1);\beta;\ \sigma+1,\ \tfrac{1}{2};\ x,\ q\ ]$$

$$-2\sqrt{q}\; \Gamma(\tfrac{1}{2}(\eta+\sigma+\delta+v+2)) \quad \Psi_1\ [\tfrac{1}{2}(\eta+\sigma+\delta+v+2);\beta;\ v+1,\ {}^{3}/_{2};\ x,\ q\ ]\ \Big\}$$

$$= \Big\{ \Gamma(\tfrac{1}{2}(\eta+\sigma+\delta+v+1)) \; F^{1:1;0}_{0:1;1}\left[\begin{array}{l} \tfrac{1}{2}(\eta+\sigma+\delta+v+1) : \beta\ ;\text{---}; \\ \overline{\qquad\qquad\qquad} : v+1;\ \tfrac{1}{2}; \end{array} \; x,\ q\right]$$

$$-2\sqrt{q}\; \Gamma(\tfrac{1}{2}(\eta+\sigma+\delta+v+2)) \; F^{1:1;0}_{0:1;1}\left[\begin{array}{l} \tfrac{1}{2}(\eta+\sigma+\delta+v+2) : \beta\ ;\text{---}; \\ \overline{\qquad\qquad\qquad} : v+1;{}^{3}/_{2}; \end{array} \; x,\ q\right] \Big\}. \qquad (7.3.24)$$

If in (7.3.18), we set $\alpha = 2\beta = \eta+\sigma+\delta+v$, replace y and x by y^2 and x^2 respectively and apply the result (7.3.7), we obtain

$$F^{2:2;0}_{0:3;1}\left[\begin{array}{l} \tfrac{1}{2}\alpha+\tfrac{1}{2},\tfrac{1}{2}\alpha:\ \tfrac{1}{2}(\sigma+\delta+2),\ \tfrac{1}{2}(\sigma+\delta+1);\text{—};\\ \text{———}:\ \sigma+\delta+1,\sigma+1,\delta+1\quad;\ v+1; \end{array} 4y^2,\ x^2\right]$$

$$= (1+2y+x)^{-\alpha}\ F_A^{(3)}\ [\alpha,\ \sigma+\tfrac{1}{2},\ \delta+\tfrac{1}{2},\ v+\tfrac{1}{2};\ 2\sigma+1,\ 2\delta+1,\ 2v+1;$$

$$\frac{2y}{(1+2y+x)},\ \frac{2y}{(1+2y+x)},\ \frac{2x}{(1+2y+x)}\ ]. \qquad (7.3.25)$$

On setting η=a-σ-δ-v, β=a/2, σ=d-1, δ=b-½, v=c-½ in (7.3.5), replacing z, y and x by $\frac{4z}{(1-y-x)^2}$, $\frac{y^2}{(1-y-x)^2}$ and $\frac{x^2}{(1-y-x)^2}$ respectively and comparing with the result of Khan and Pathan [48,p.88(4.4)]

$$H_4^{(p)}\left[a;\ b,\ c;\ d,\ 2b,\ 2c;\ x,\ 2y,\ 2z\ \right] = (1-y-z)^{-a}$$

$$F_c^{(3)}\left[\tfrac{1}{2}a,\ \tfrac{1}{2}a+\tfrac{1}{2};\ d,\ b+\tfrac{1}{2},\ c+\tfrac{1}{2};\ 4x/(1-y-z)^2,\ y^2/(1-y-z)^2,\ z^2/(1-y-z)^2\right],$$

(7.3.26)

we get

$$\left(1-\frac{y^2}{4z}\right)^{b+d-\frac{1}{2}} F^{(3)}\left[\begin{array}{l} -::b+d-\tfrac{1}{2},\ b+\tfrac{1}{2};-;(a+1)/2,a/2:\text{———};\ \text{—};\text{—};\\ -::\text{———};-;\text{———}:b+d-\tfrac{1}{2},d,b+\tfrac{1}{2};b+\tfrac{1}{2};c+\tfrac{1}{2}; \end{array}\right.$$

$$\left.\left(\frac{4z}{(1-y-x)^2}\right)\left(1-\frac{y}{4z}\right)^2,\ \frac{y^2}{4z},\ \frac{x^2}{(1-y-x)^2}\right]$$

$$= (1-y-x)^2\ H_4^{(p)}\ [a;\ b,\ c;\ d,\ 2b,\ 2c;\ x,\ 2y,\ 2z\ ], \qquad (7.3.27)$$

where $H_4^{(p)}$ is Horn's function [48,p.85(1.1)] defined by equation (1.12.6).

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Dear Drs, M.A.PATHAN and M.G.BIN SAAD

Concerning your manuscript "**On Double Genereting Functions of Single Hypergeometric Polynomials**", submitted for pubblication to *Rivista di Matematica della Università di Parma* , and received by our office on august 17, 1999, I am glad to inform you that the paper has been accepted for publication.

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